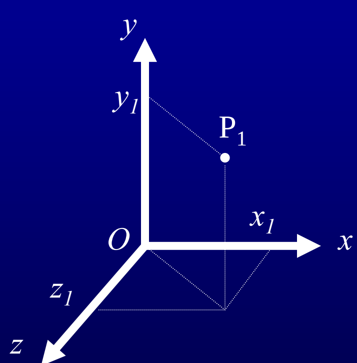
Basics

1

Points



- three numbers suffice
- but specify where you measure from!

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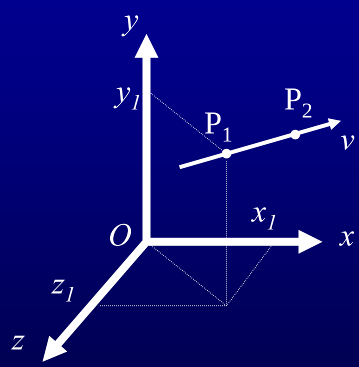
Directions



- From P_1 towards P_2

$P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2)$

vector

$v = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

length

$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

normalized – vector

$(\frac{x_2 - x_1}{l}, \frac{y_2 - y_1}{l}, \frac{z_2 - z_1}{l})$

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Vector operations

- Dot product
  - result is a scalar (one number)
  - cosine of the angle between the vectors times the product of the lengths of the vectors
  - if unit length vectors, dot product is cosine of angle between the two directions
  - commutative

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = \sqrt{x_1{}^2 + y_1{}^2 + z_1{}^2}\sqrt{x_2{}^2 + y_2{}^2 + z_2{}^2}\cos\theta$$

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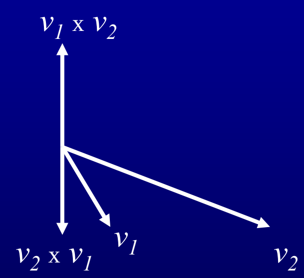
Vector operations

- Cross-product
  - result is a vector
    - perpendicular to both operands
    - length is lengths times sine of angle
  - normal of operands plane
  - not commutative



$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2)$$

$$\|v_1 \times v_2\| = \|v_1\| \cdot \|v_2\| \cdot |\sin\theta|$$

$$v_1 \times v_2 = -v_2 \times v_1$$

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Geometry

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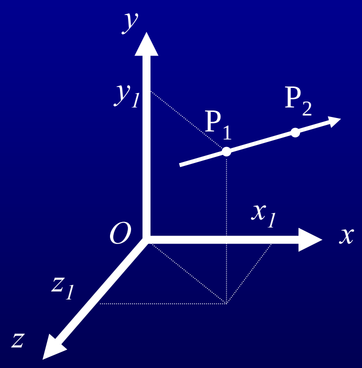
Line



$P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2)$

line

$$\begin{cases} P = P_1 + (P_2 - P_1)t \\ t \in (-\infty, \infty) \end{cases}$$

segment

$$\begin{cases} P = P_1 + (P_2 - P_1)t \\ t \in [0,1] \end{cases}$$

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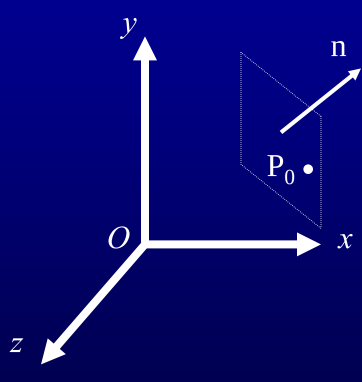
Planes (point and normal)



$P_0(x_0, y_0, z_0), n(n_x, n_y, n_z)$

plane

$n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0$

$(P - P_0)n = 0$

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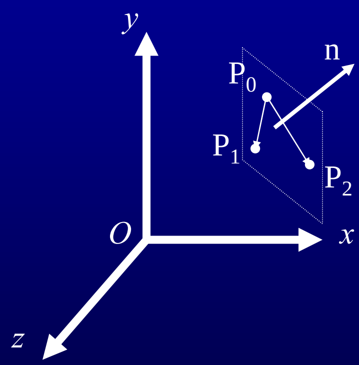
Planes (3 points)



$$P_0(x_0, y_0, z_0), P_1(x_1, y_1, z_1), P_0(x_2, y_2, z_2)$$

normal

$$n = (P_1 - P_0) \times (P_2 - P_0)$$

plane

$$(P - P_0)n = 0$$

- Normal direction given by enumeration order

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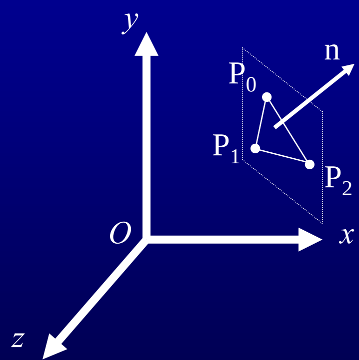
Triangles (3 points)



$$P_0(x_0, y_0, z_0), P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2)$$

normal

$$n = (P_1 - P_0) \times (P_2 - P_0)$$

plane

$$(P - P_0)n = 0$$

triangle

$$\begin{cases} (P - P_0)n = 0 \\ Sidedness(P, P_0, P_1) = Sidedness(P_2, P_0, P_1) \\ Sidedness(P, P_1, P_2) = Sidedness(P_0, P_1, P_2) \\ Sidedness(P, P_2, P_0) = Sidedness(P_1, P_2, P_0) \end{cases}$$

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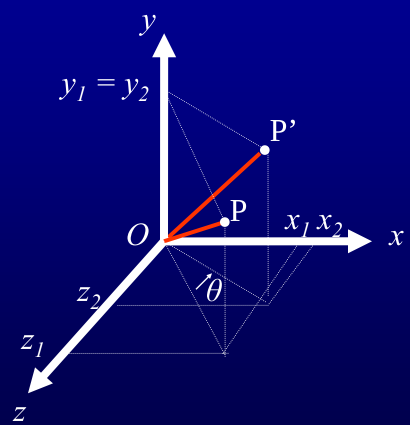
Transformations

Translations

- Points
  - $P_1(x_1, y_1, z_1)$ moves to $P_2(x_2, y_2, z_2)$
  - translation amount $(x_2-x_1, y_2-y_1, z_2-z_1)$
- Vectors are invariant to translations
  - "up" means "up" no matter where you are
- Segments and triangles are translated by translating their defining points

Rotations



- Rotation about y axis

$$x_2 = x_1 \cos\theta + z_1 \sin\theta$$
$$y_2 = y_1$$
$$z_2 = -x_1 \sin\theta + z_1 \cos\theta$$

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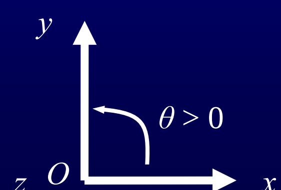
Positive rotations

- Convention
  - in right-handed coordinate systems
  - look down axis to rotate about (towards origin)
  - counterclockwise rotation transforms one positive axis into the other



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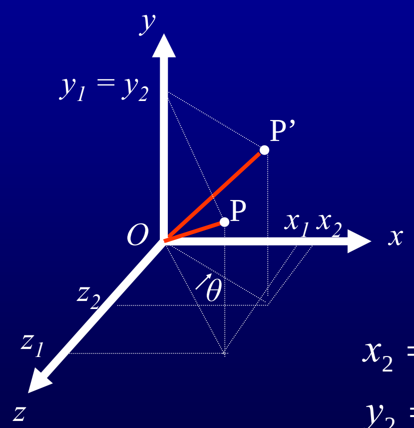
Rotations



- Rotation about y axis
  - rotation by positive amount
  - rotation of point P
  - rotation of vector OP
  - segments and triangles are rotated by rotating points

$$x_2 = x_1 \cos\theta + z_1 \sin\theta$$
$$y_2 = y_1$$
$$z_2 = -x_1 \sin\theta + z_1 \cos\theta$$

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Matrix notation

- Rotation about y axis

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1 \cos\theta + z_1 \sin\theta \\ y_1 \\ -x_1 \sin\theta + z_1 \cos\theta \end{bmatrix}$$

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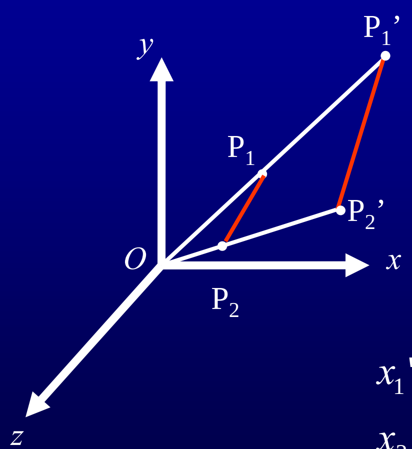
Scaling



- Example of scaling a segment

$$x_1' = x_1 s_x, y_1' = y_1 s_y, z_1' = z_1 s_z$$

$$x_2' = x_2 s_x, y_2' = y_2 s_y, z_2' = z_2 s_z$$

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Combinations of xforms

- A chain of transformations
  - matrices are multiplied
  - matrix multiplication is associative
  - matrix multiplication is NOT commutative
  - equivalent matrix

$$X_1 \cdot X_2 \cdot \ldots \cdot X_n \cdot P = P'$$

$$(X_1 \cdot X_2 \cdot \ldots \cdot X_n) \cdot P = P'$$

$$X \cdot P = P'$$

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Combinations of xforms

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_{14} \\ r_{21} & r_{22} & r_{23} & t_{24} \\ r_{31} & r_{32} & r_{33} & t_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sqrt{r_{11}^2 + r_{21}^2 + r_{31}^2} = 1$$

$$\sqrt{r_{12}^2 + r_{22}^2 + r_{32}^2} = 1$$

$$\sqrt{r_{13}^2 + r_{23}^2 + r_{33}^2} = 1$$

$$(r_{11}, r_{21}, r_{31}) \cdot (r_{12}, r_{22}, r_{32}) = 0$$

$$(r_{12}, r_{22}, r_{32}) \cdot (r_{13}, r_{23}, r_{33}) = 0$$

$$(r_{13}, r_{23}, r_{33}) \cdot (r_{11}, r_{21}, r_{31}) = 0$$

- Rigid body
  - any combination of rotations and translations
  - R (r_{11}, …, r_{33}) is orthographic
    - columns (and lines) are mutually perpendicular unit vectors
  - preserves lengths and angles

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Combinations of xforms

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_{14} \\ r_{21} & r_{22} & r_{23} & t_{24} \\ r_{31} & r_{32} & r_{33} & t_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Affine transform
  - any combination of rotations, translations and scalings
  - does not preserve angles and lengths
  - does preserve parallelism of lines

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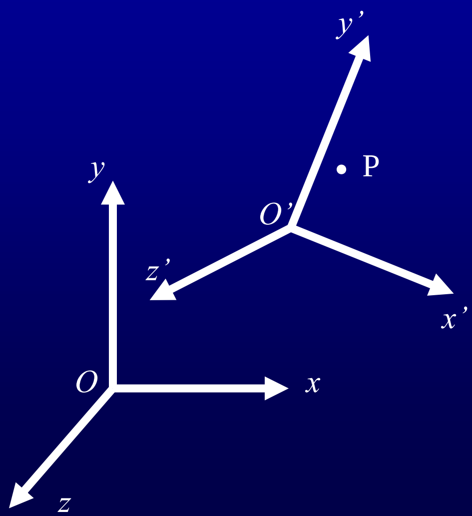
Change of coordinate system



- Given
  - point P(x, y,z)
  - new coordinate system
    - unit vectors *x'*, *y'*, *z'*
    - origin *O'*
- Find P' (coordinates of point P in new coordinate system)

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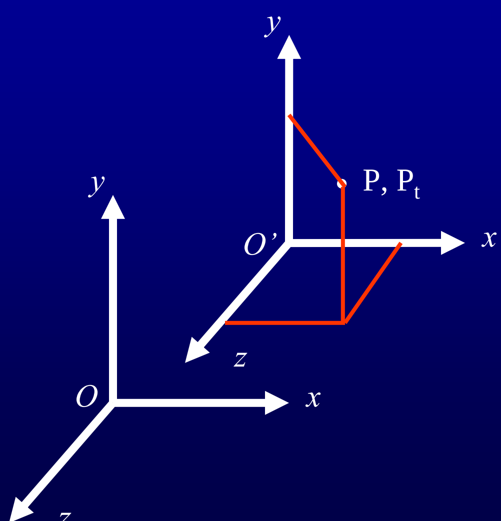
Change of coordinate system



- First translate to new origin

$$P_t = P - O'$$

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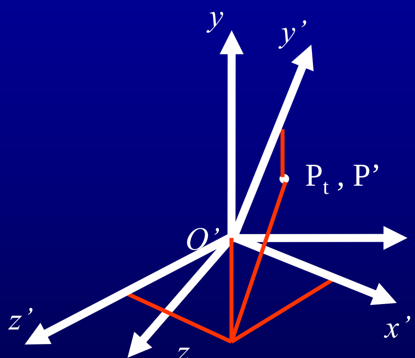
Change of coordinate system



- Then rotate old axes into new axes
  - compute components of vector OP along new axes

$$P' = \begin{bmatrix} x'_x & x'_y & x'_z \\ y'_x & y'_y & y'_z \\ z'_x & z'_y & z'_z \end{bmatrix} \cdot \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}$$

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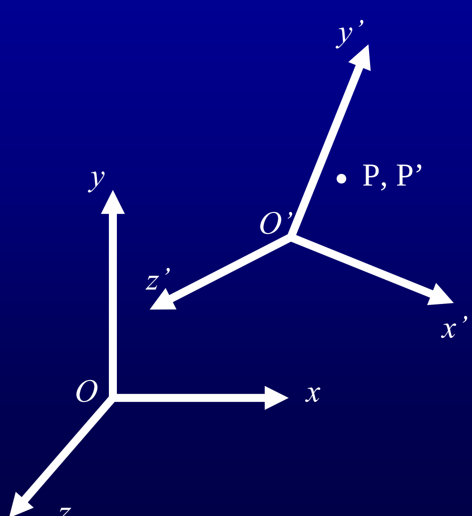
Change of coordinate system



$$P' = \begin{bmatrix} x'_x & x'_y & x'_z \\ y'_x & y'_y & y'_z \\ z'_x & z'_y & z'_z \end{bmatrix} \cdot \begin{bmatrix} x - O'_x \\ y - O'_y \\ z - O'_z \end{bmatrix}$$

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Rotation about an arbitrary axis

- Given
  - axis with origin O_a and direction a
  - point P, angle θ
- Rotate P θ degrees about the axis (O_a, a)
  - compute P^r which is the rotated point P

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Step 1

- Create new coordinate system with origin O_a and a as one of its axes
  - using axis $x(1, 0, 0)$ set $b = (x \times a)$; normalize b
  - set $c = a \times b$; normalize c
  - (O_a, a, b, c) is a new coordinate system
  - note: one needs to make sure that the auxiliary axis (in this case x) is not aligned with a; one solution is to consider x and y and pick the one that has the smaller dot product with a (ignoring the sign) since that axis is closer to being perpendicular to a.

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Steps 2-4

- Step 2: Transform *P* to the new coordinate system, P->P’
- Step 3: Rotate about *a* (first axis), P’->P’’
- Step 4: Transform back to original coordinate system, P’’->P^r

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