Curves, Splines, Patches, Basis

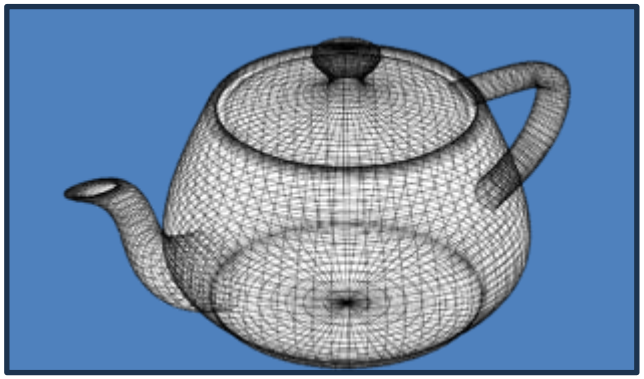
CS535

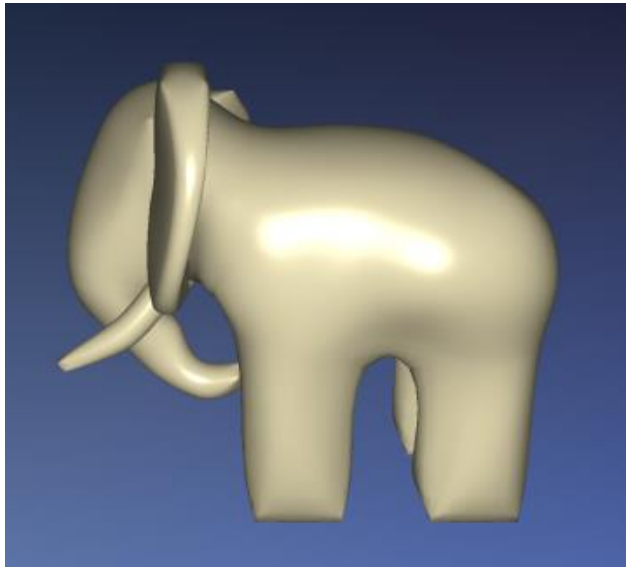
Daniel G. Aliaga
Department of Computer Science
Purdue University

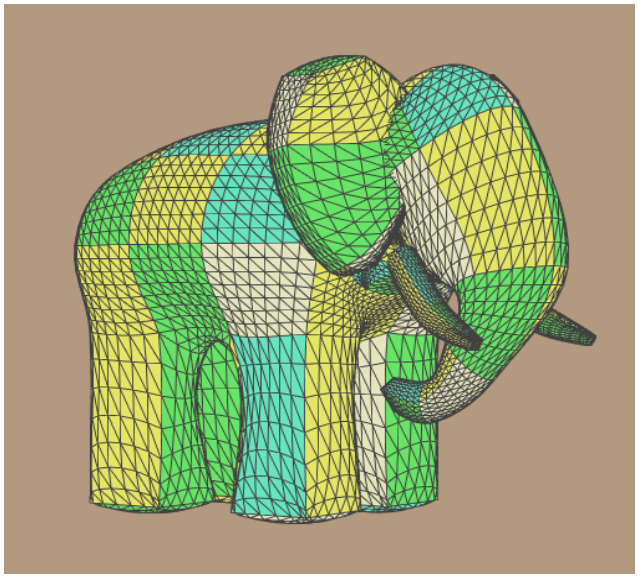
[some slides based on Bruce Cohen and David Sklar, and Jack van Wijk]

Planar (Triangular) Patches vs Curved Patches

Planar (Triangular) Patches vs Curved Patches

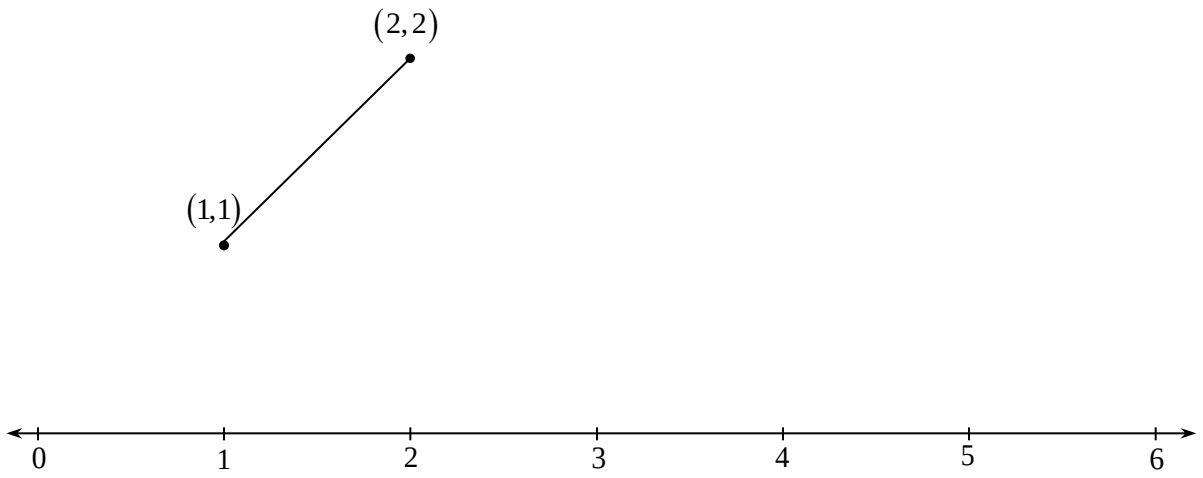






Connecting the Dots

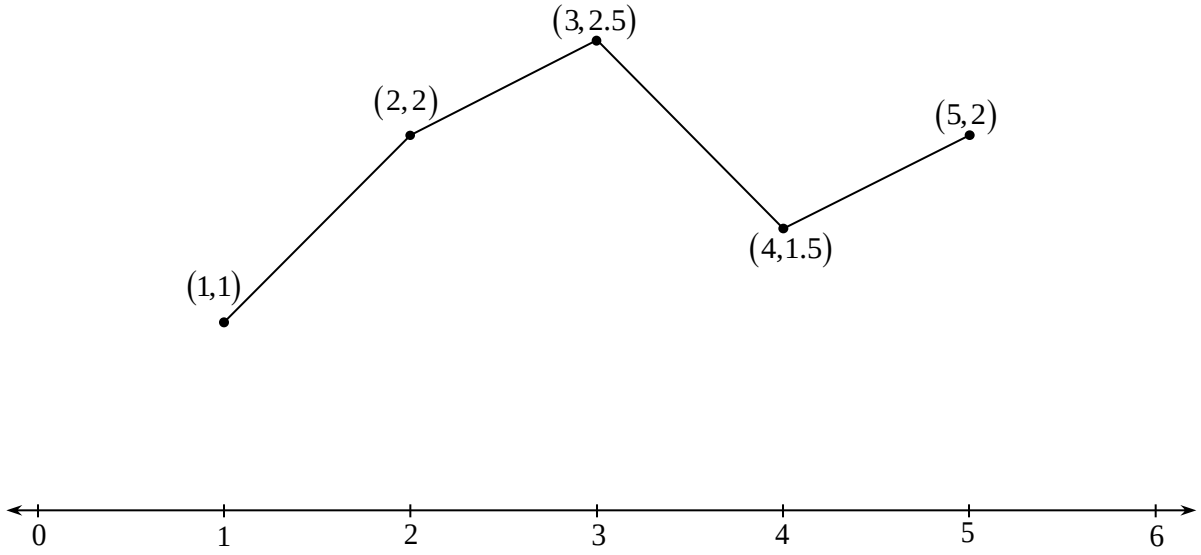
What is the formula for this line?



$$f(x) = x; f(x) = (1 - x) * (1,1) + x * (2,2)$$

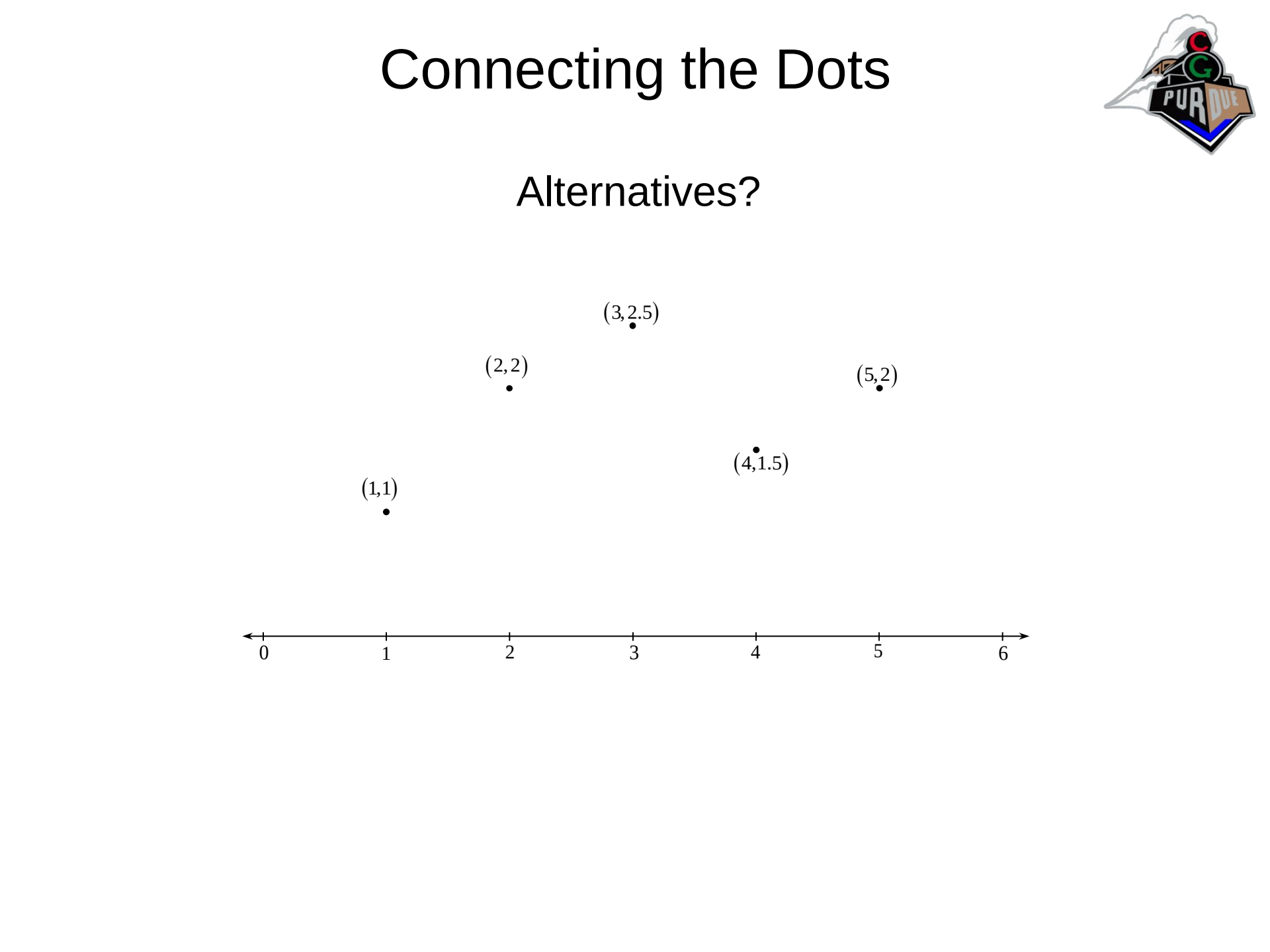
Connecting the Dots

Write a formula for a piecewise linear function that interpolates five given data points



$$f(x) = \begin{cases} x & \text{if } 1 \le x \le 2 \\ \frac{1}{2}x + 1 & \text{if } 2 \le x \le 3 \\ -x + \frac{11}{2} & \text{if } 3 \le x \le 4 \\ \frac{1}{2}x - \frac{1}{2} & \text{if } 4 \le x \le 5 \end{cases}$$

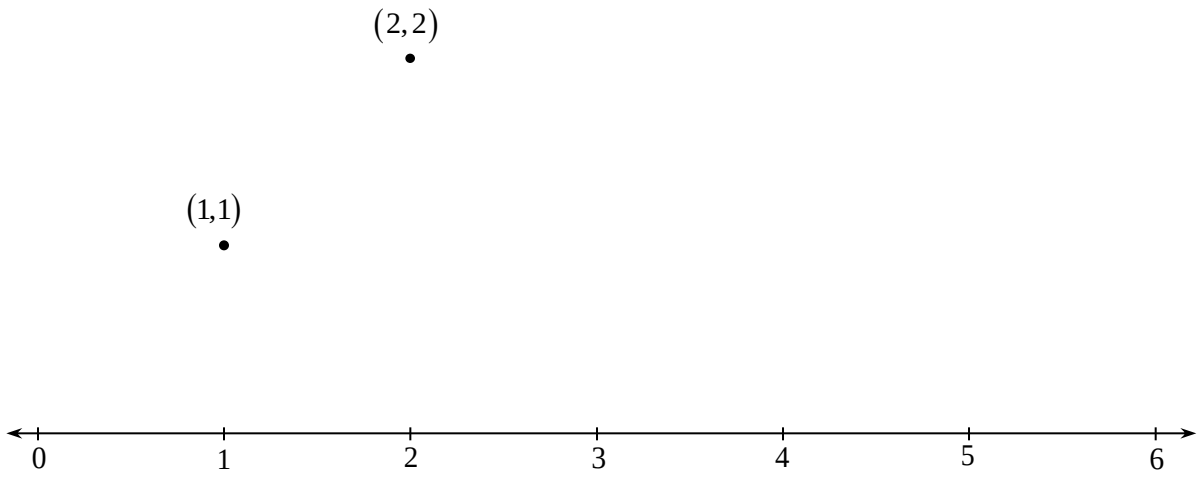
Connecting the Dots



Alternatives?

Connecting the Dots

Interpolating Polynomial



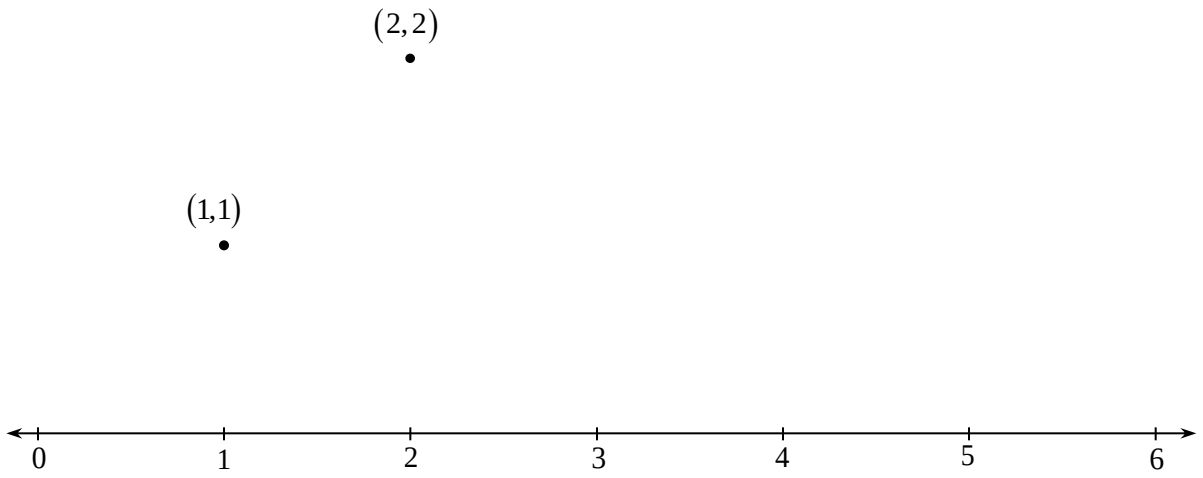
$f(1) = 1$ and $f(2) = 2$

$h_0(x) = \frac{x - x_1}{x_0 - x_1}$ is 1 when $x = 1$ and is 0 when $x = 2$

$h_1(x) = \frac{x - x_0}{x_1 - x_0}$ is 1 when $x = 2$ and is 0 when $x = 1$

Connecting the Dots

Interpolating Polynomial



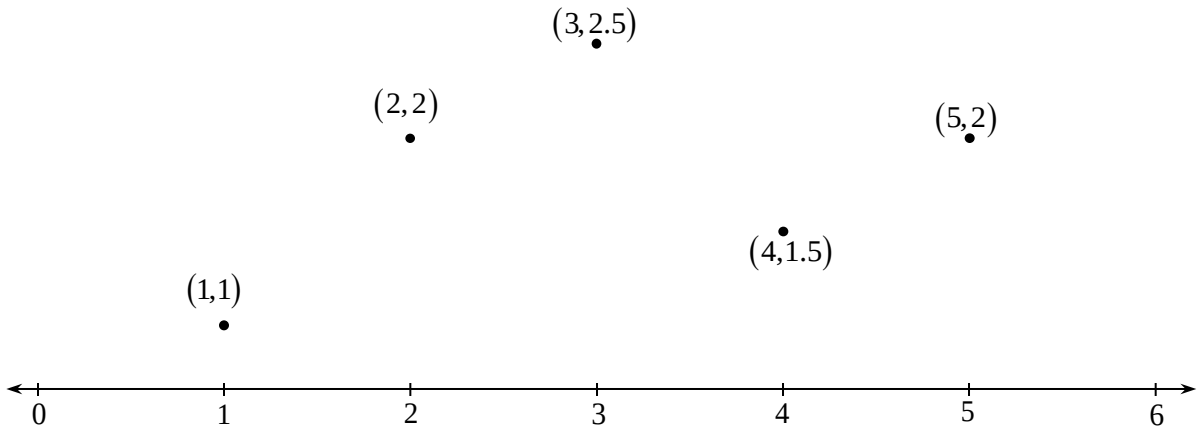
$f(1) = 1$ and $f(2) = 2$

$$f(x) = y_0 h_0(x) + y_1 h_1(x)$$

thus $f(x_0) = y_0$ and $f(x_1) = y_1$

Connecting the Dots

Interpolating Polynomial



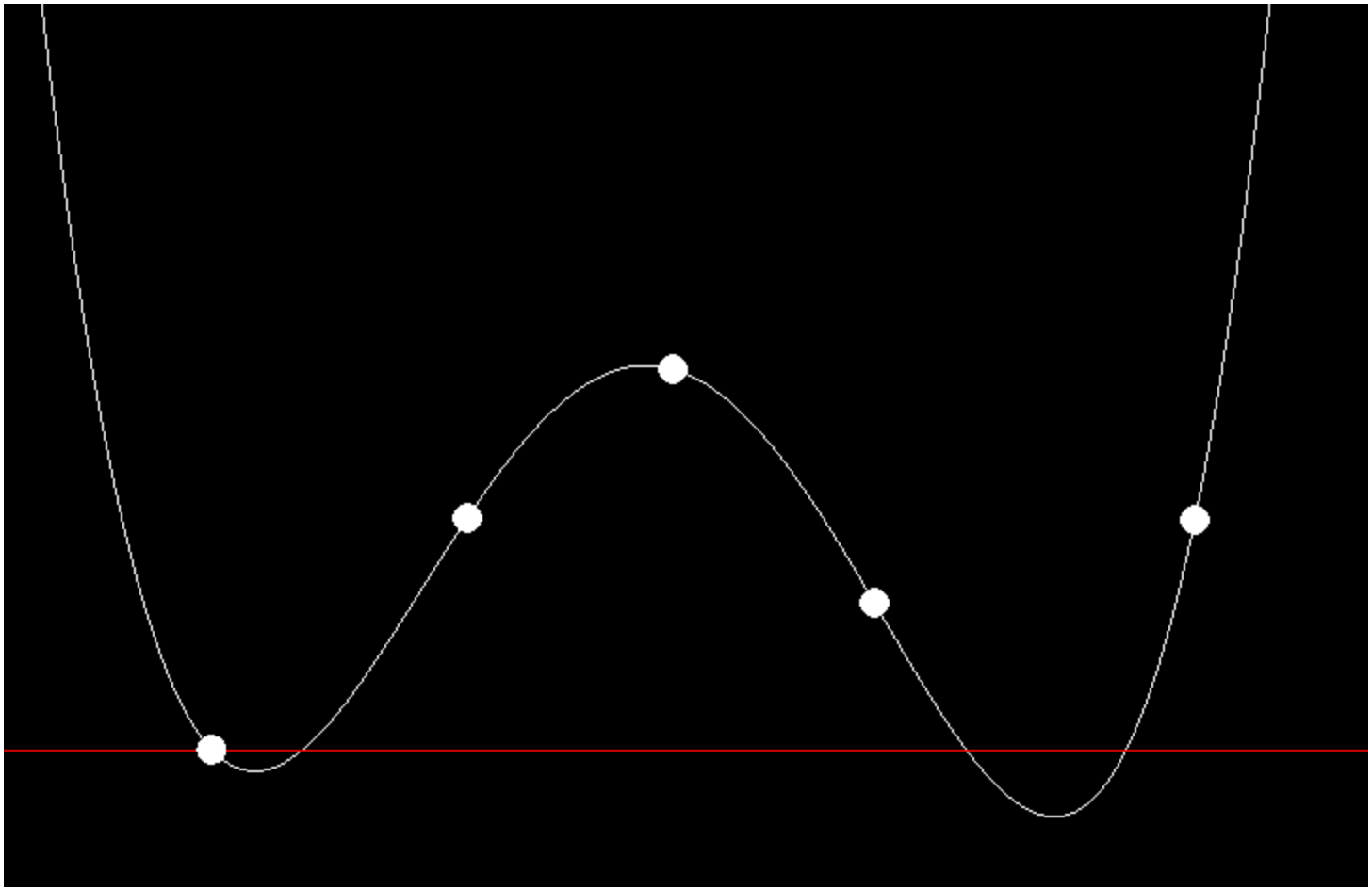
$$\phi_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

Lagrange Interpolation Basis

$$\phi_i(x_l) = 1 \; when \; l = i, else \; 0$$

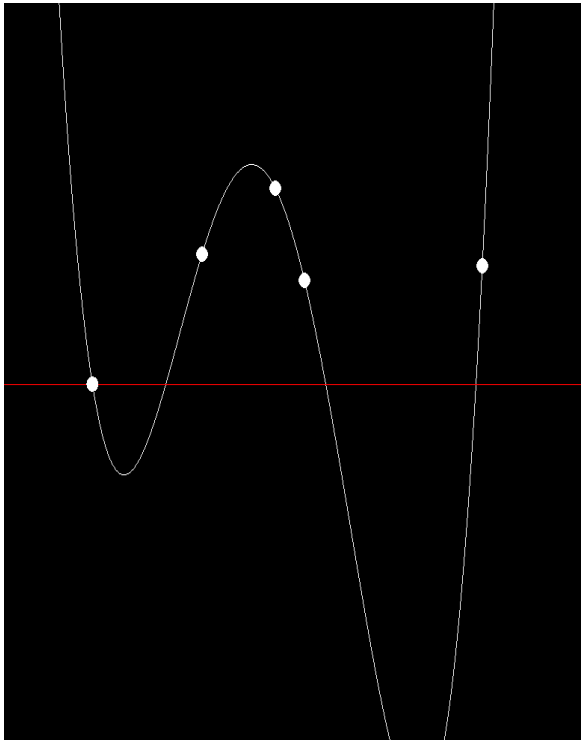
$$f(x) = \sum_i y_i \phi_i(x)$$

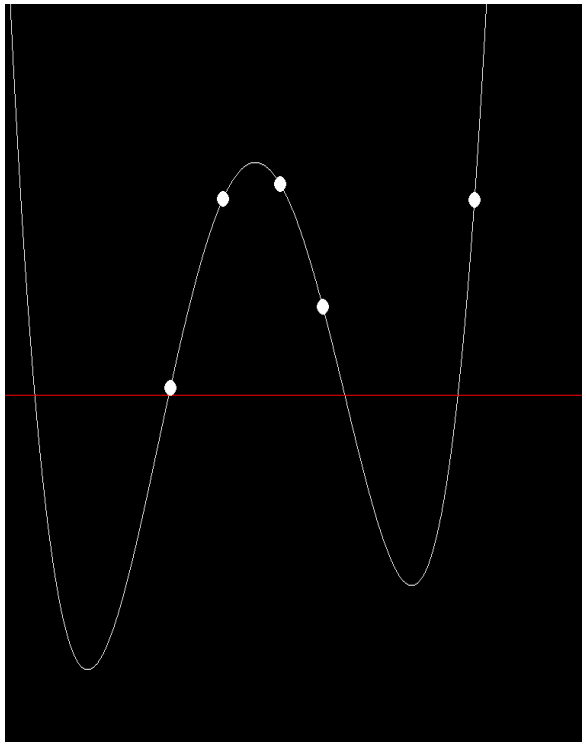
Lagrange Polynomial

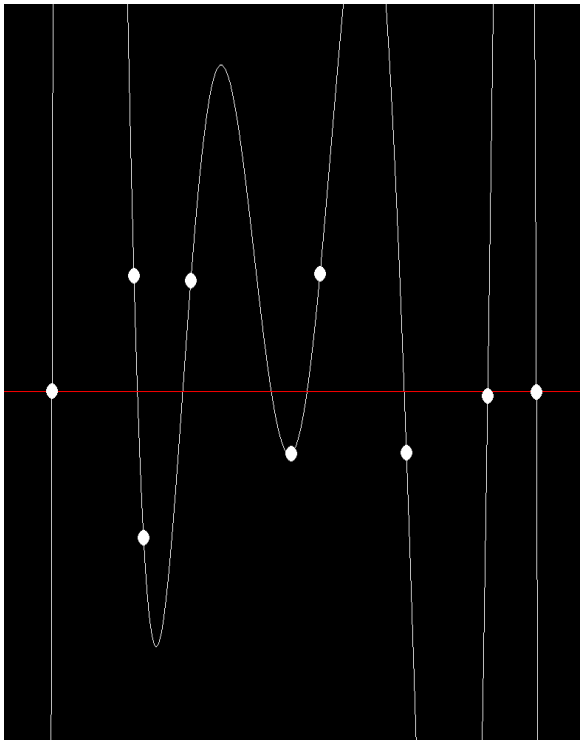


https://l-e-webb.github.io/interpolate/webdist/index.html

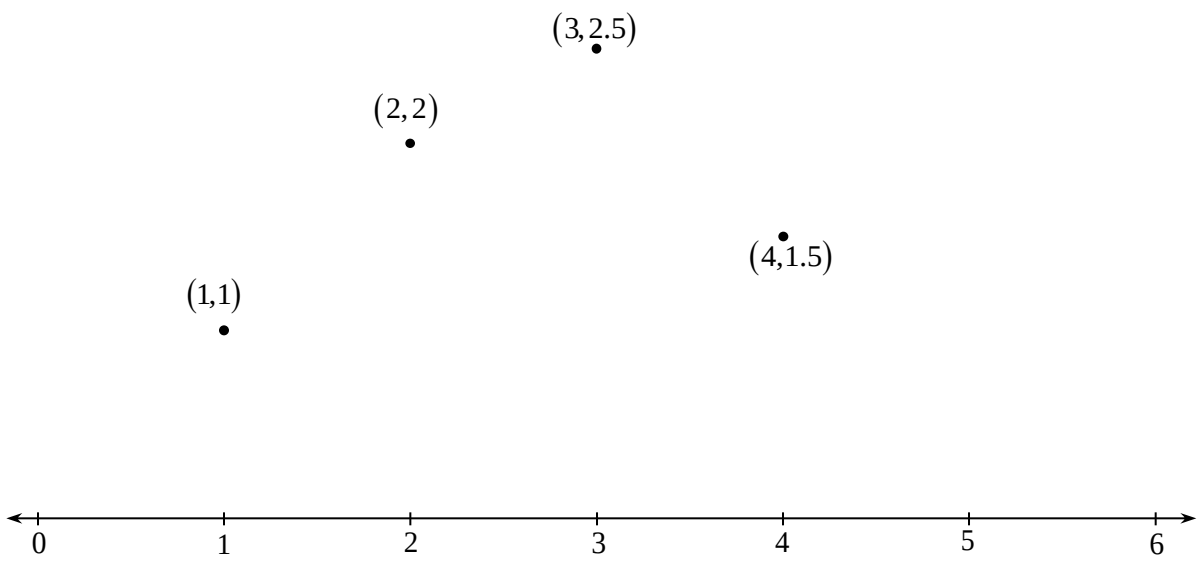
But not very stable…



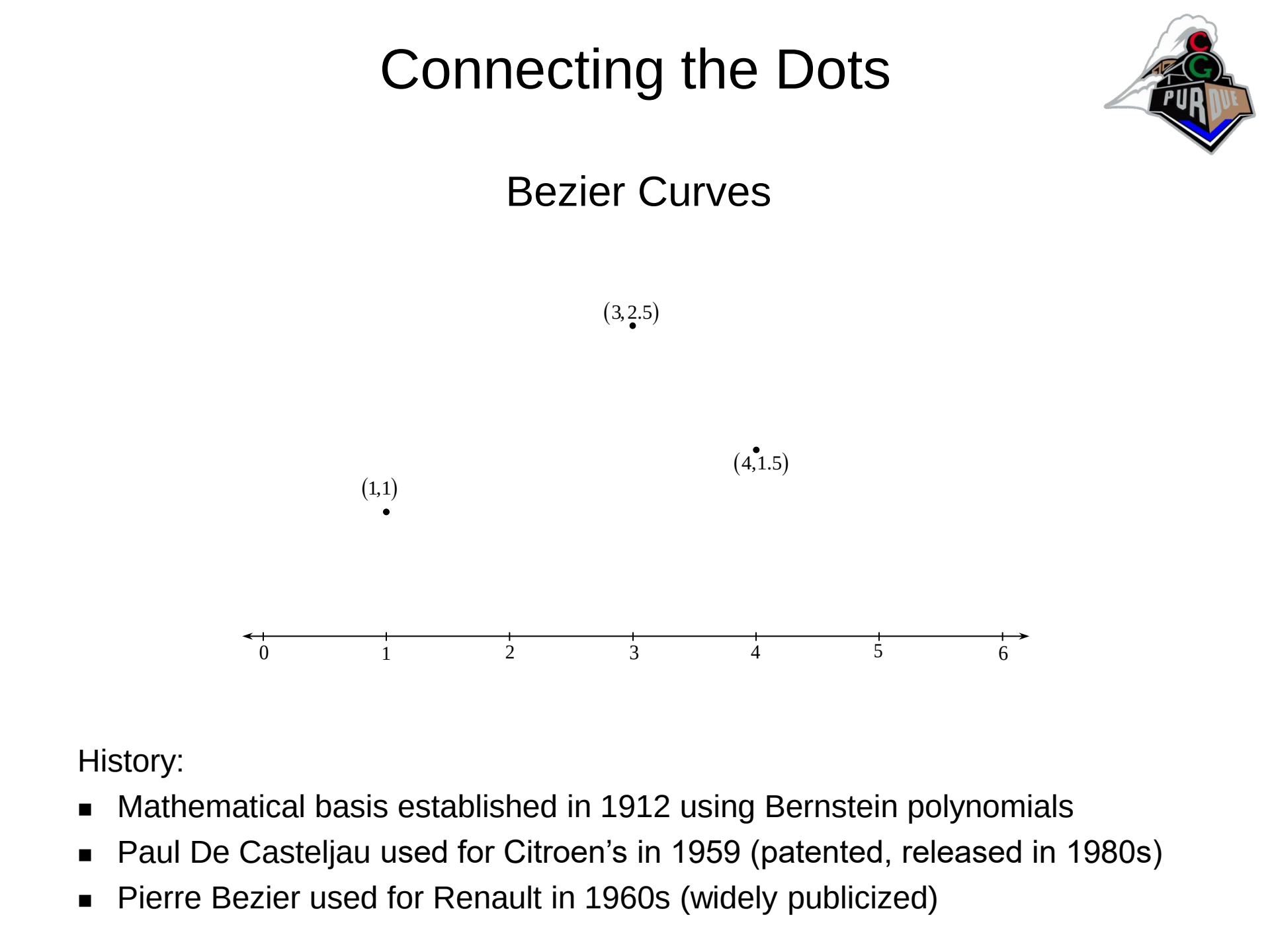




Connecting the Dots



Connecting the Dots



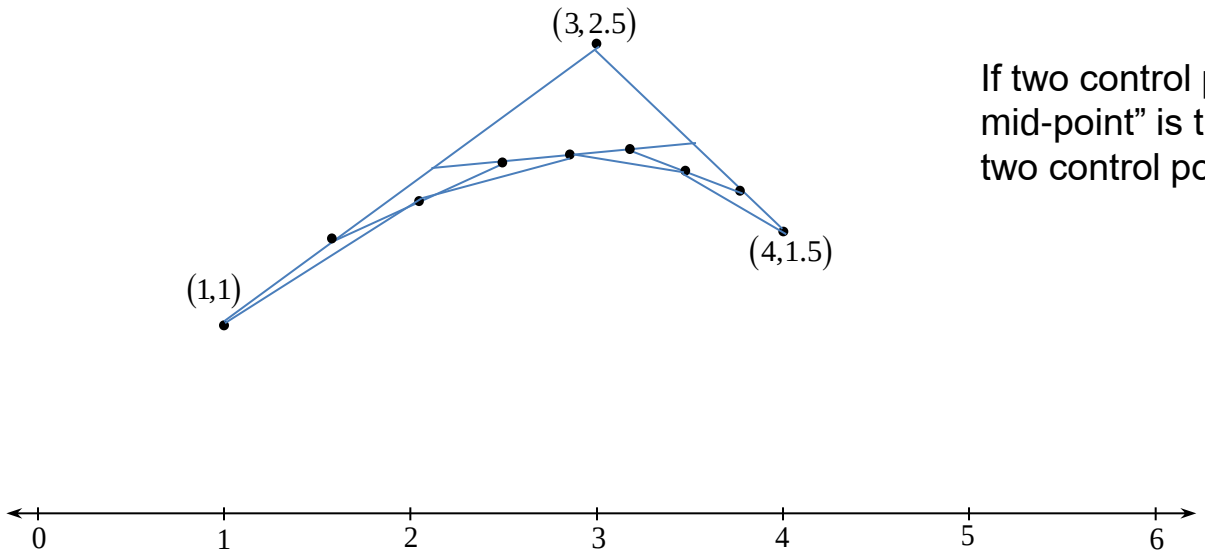
Bezier Curves

History:

- Mathematical basis established in 1912 using Bernstein polynomials
- Paul De Casteljau used for Citroen's in 1959 (patented, released in 1980s)
- Pierre Bezier used for Renault in 1960s (widely publicized)

Connecting the Dots

Bezier Curves



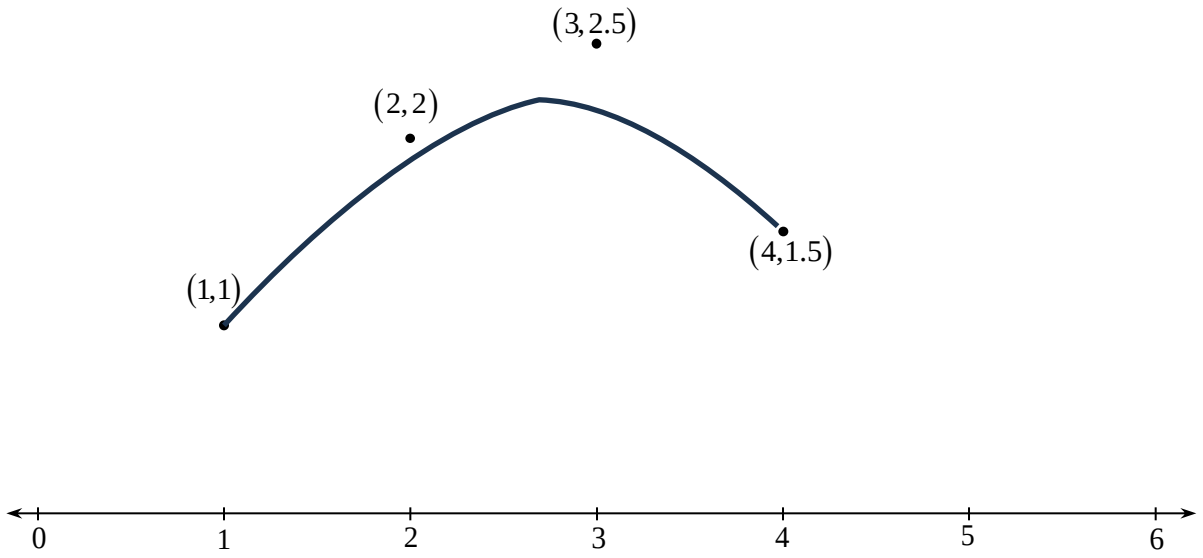
If two control points, then “first mid-point” is the midpoint of the two control points…

History:

- Mathematical basis established in 1912 using Bernstein polynomials
- Paul De Casteljau used for Citroen’s in 1959 (patented, released in 1980s)
- Pierre Bezier used for Renault in 1960s (widely publicized)

Connecting the Dots

Bezier Curves



Types ($[0 \leq t \leq 1]$):

Linear: $B(t) = (1-t)p_0 + tp_1$

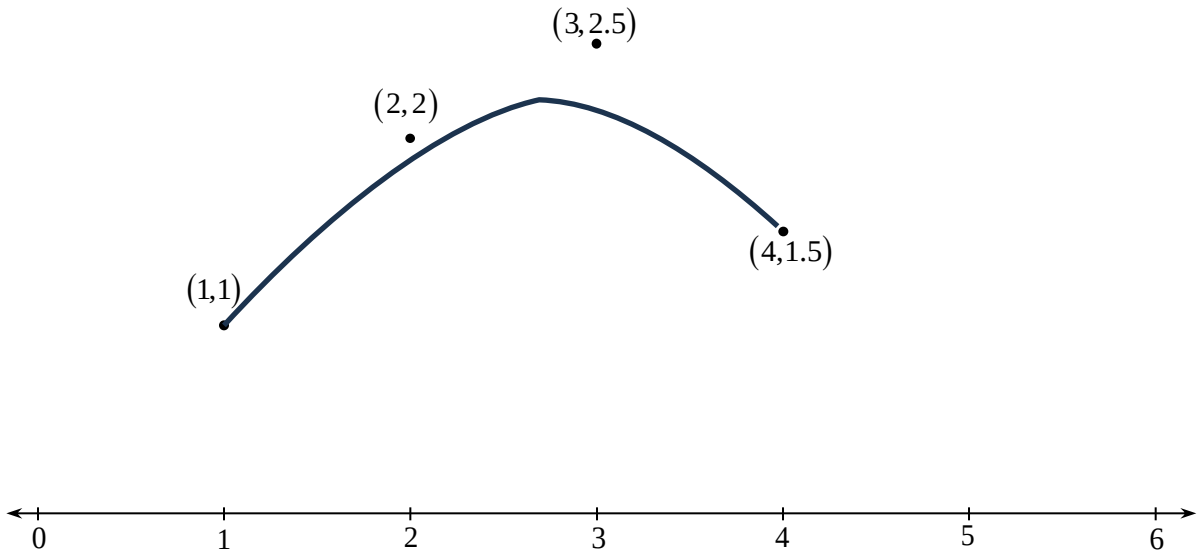
Quadratic: $B(t) = p_1 + (1-t)^2(p_0 - p_1) + t^2(p_2 - p_1)$

Cubic: $B(t) = (1-t)B_{p_0,p_1,p_2} + tB_{p_1,p_2,p_3}(t)$

$B(t) = (1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t)t^2 p_2 + t^3 p_3$

Connecting the Dots

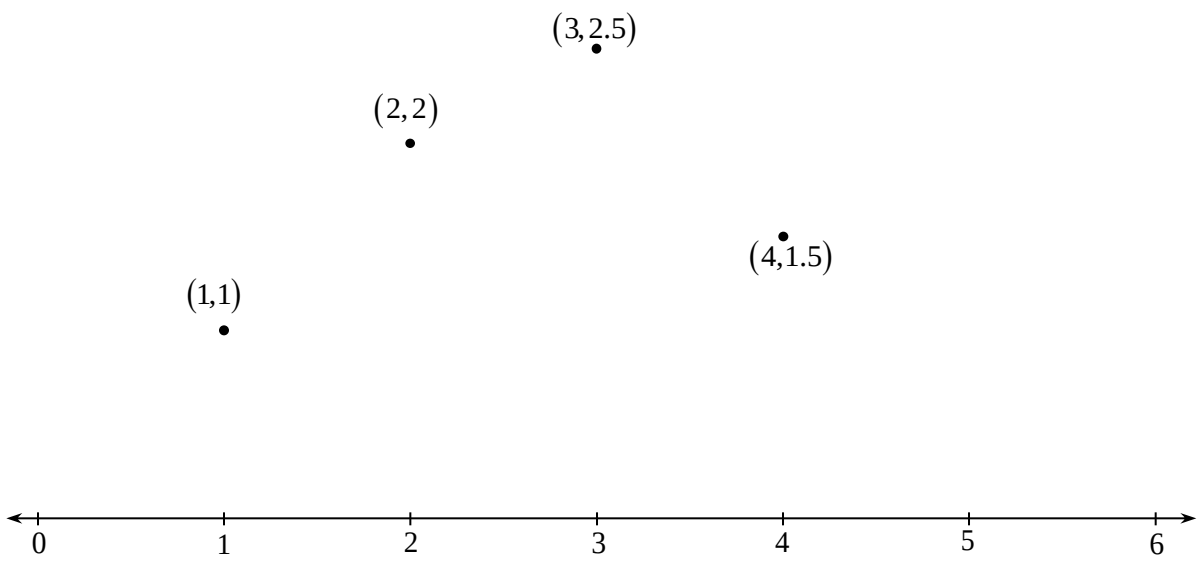
Bezier Curves



Properties:

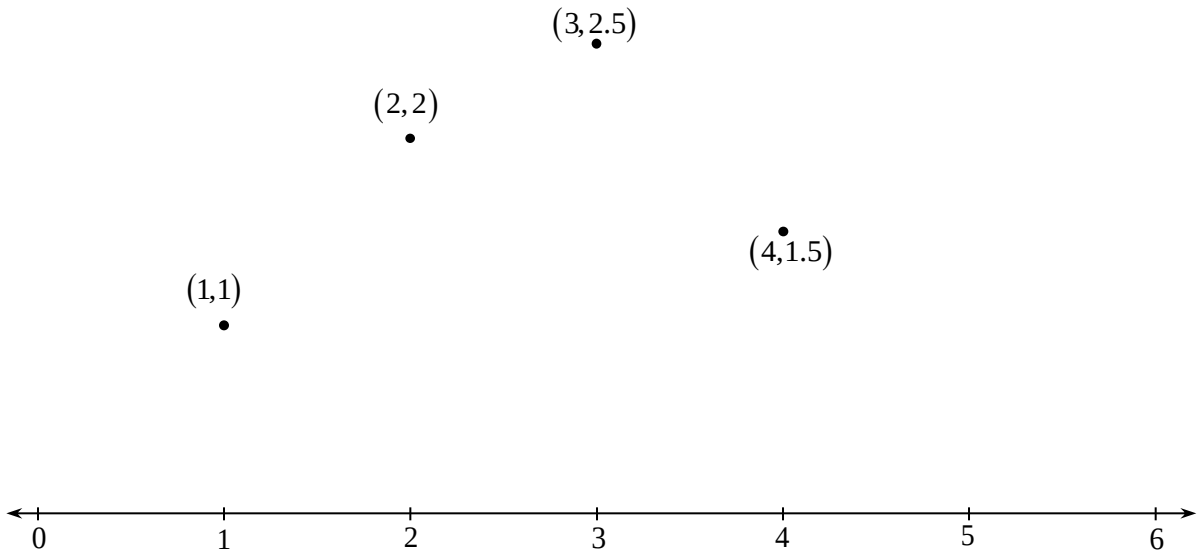
- Curve begins at p_0 and ends at p_3 (and does not necessarily pass thru p_1 and p_2)
- Start and end is tangent to first/last section
- Curves have the variation diminishing property – do not undulate more than its control points

Connecting the Dots



Connecting the Dots

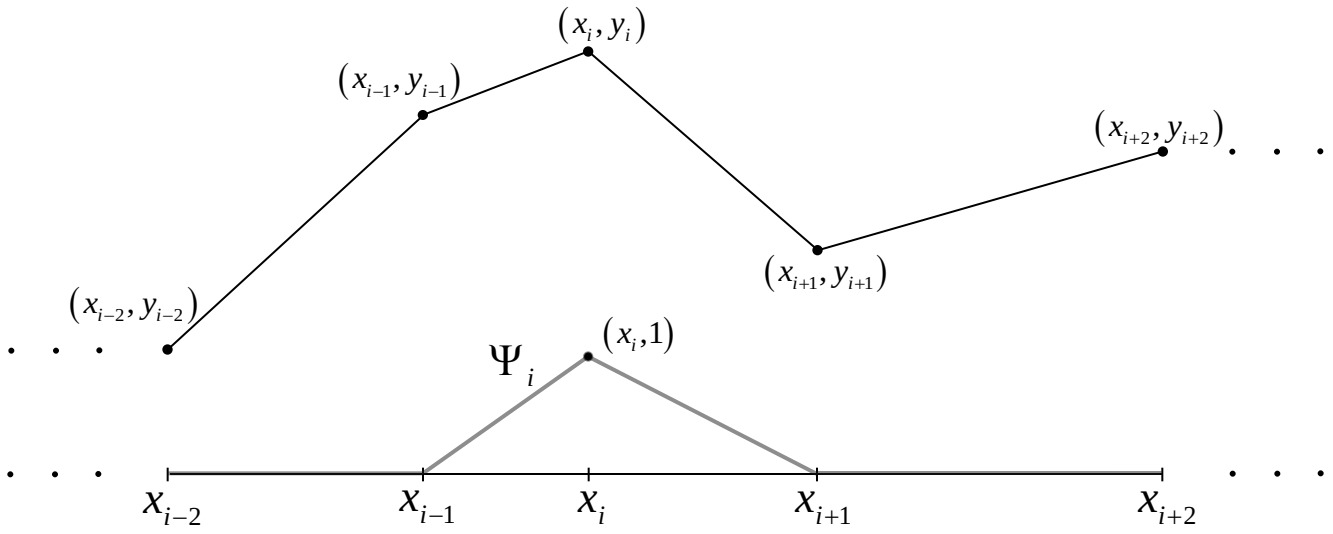
B-splines (Basis Splines)



A spline uses a set of basis functions to express the line as a sum of weighted components (not necessarily Bernstein Polynomials:

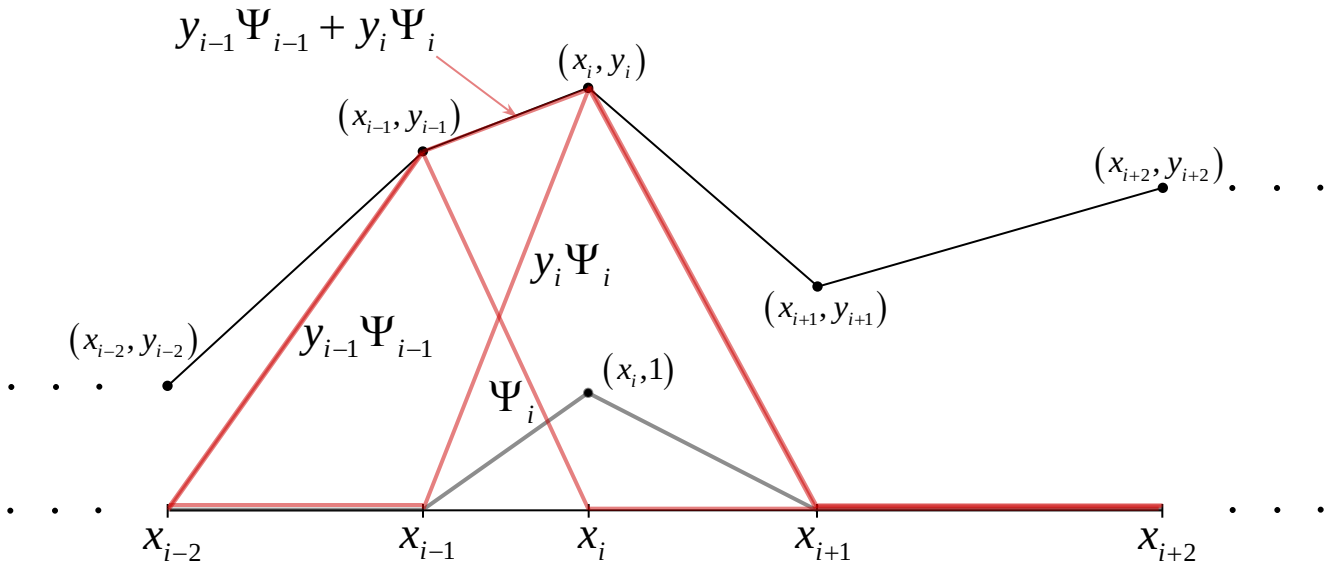
$$f(x) = y_1\Psi_1(x) + y_2\Psi_2(x) + \cdots + y_n\Psi_n(x) = \sum_{i=1}^{n} y_i\Psi_i(x) \quad \text{, for all } x_1 \le x \le x_n$$

B-splines:
With linear basis functions…



A closer look at a linear combination of basis functions

$$f(x) = y_1\Psi_1(x) + y_2\Psi_2(x) + \cdots + y_n\Psi_n(x) = \sum_{i=1}^{n} y_i\Psi_i(x)$$

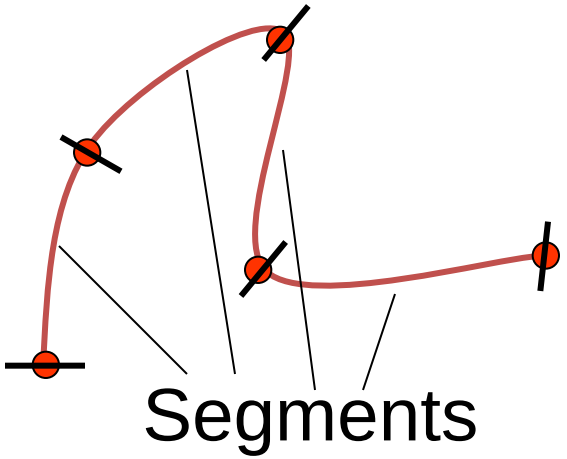


Cubic Splines

Sequence of segments defined by :

$$\mathbf{P}(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} = \begin{pmatrix} a_x u^3 + b_x u^2 + c_x u + d_x \\ a_y u^3 + b_y u^2 + c_y u + d_y \\ a_z u^3 + b_z u^2 + c_z u + d_z \end{pmatrix}$$

with $0 \le u \le 1$.



Cubic Splines (in 2D)

Let $p(x) = ax^3 + bx^2 + cx + d$ so $p'(x) = 3ax^2 + 2bx + c$

Using the data we have

$$p(x_1) = ax_1^3 + bx_1^2 + cx_1 + d = y_1$$
$$p(x_2) = ax_2^3 + bx_2^2 + cx_2 + d = y_2$$
$$p'(x_1) = 3ax_1^2 + 2bx_1 + c = y_1'$$
$$p'(x_2) = 3ax_2^2 + 2bx_2 + c = y_2'$$

We have 4 equations in the four unknown coefficients. So we can solve for *a*, *b*, *c*, and *d*.

Cubic Spline Representation Options

$$\text{Var. 1}: \mathbf{P}(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{pmatrix}$$

$$\text{Var. 2}: \mathbf{P}(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} M_{00} & M_{01} & M_{02} & M_{03} \\ M_{10} & M_{11} & M_{12} & M_{13} \\ M_{20} & M_{21} & M_{22} & M_{23} \\ M_{30} & M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix}$$

$$\text{Var. 3}: \mathbf{P}(u) = \sum_{k=0}^{3} B_k(u)\mathbf{P}_k$$

Cubic Splines

Variant 2 :

$$\mathbf{P}(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} M_{00} & M_{01} & M_{02} & M_{03} \\ M_{10} & M_{11} & M_{12} & M_{13} \\ M_{20} & M_{21} & M_{22} & M_{23} \\ M_{30} & M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} P_{0x} & P_{0y} & P_{0z} \\ P_{1x} & P_{1y} & P_{1z} \\ P_{2x} & P_{2y} & P_{2z} \\ P_{3x} & P_{3y} & P_{3z} \end{pmatrix}$$

$$= \mathbf{U}\mathbf{M}_{\text{spline}} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix} = \mathbf{U}\mathbf{M}_{\text{spline}}\mathbf{M}_{\text{geom}}$$

Control points or
Control vectors

Matrix $\mathbf{M}_{\text{spline}}$:'translates' geometric info to coefficients

Bézier Spline Curves

Cubic Bézier spline :

$$\mathbf{P}(u) = (1-u)^3\mathbf{P}_0 + 3u(1-u)^2\mathbf{P}_1 + 3u^2(1-u)\mathbf{P}_2 + u^3\mathbf{P}_3$$

In matrix representation :

$$\mathbf{P}(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix}$$

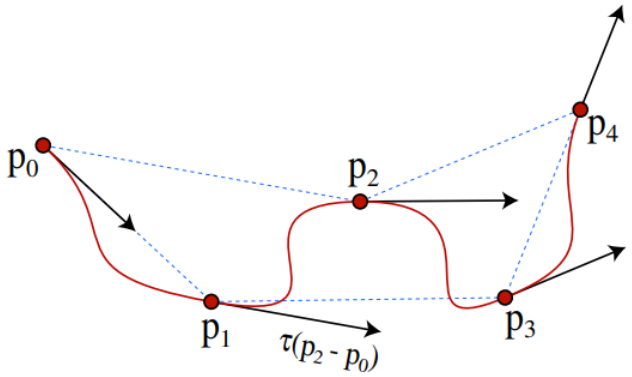
Catmull-Rom Splines

These splines "go through" the points P_0, P_1, P_2, P_3

In matrix representation:

$$\mathbf{P}(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} -\tau & 2-\tau & \tau-2 & \tau \\ 2\tau & \tau-3 & 3-2\tau & -\tau \\ -\tau & 0 & \tau & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix}$$

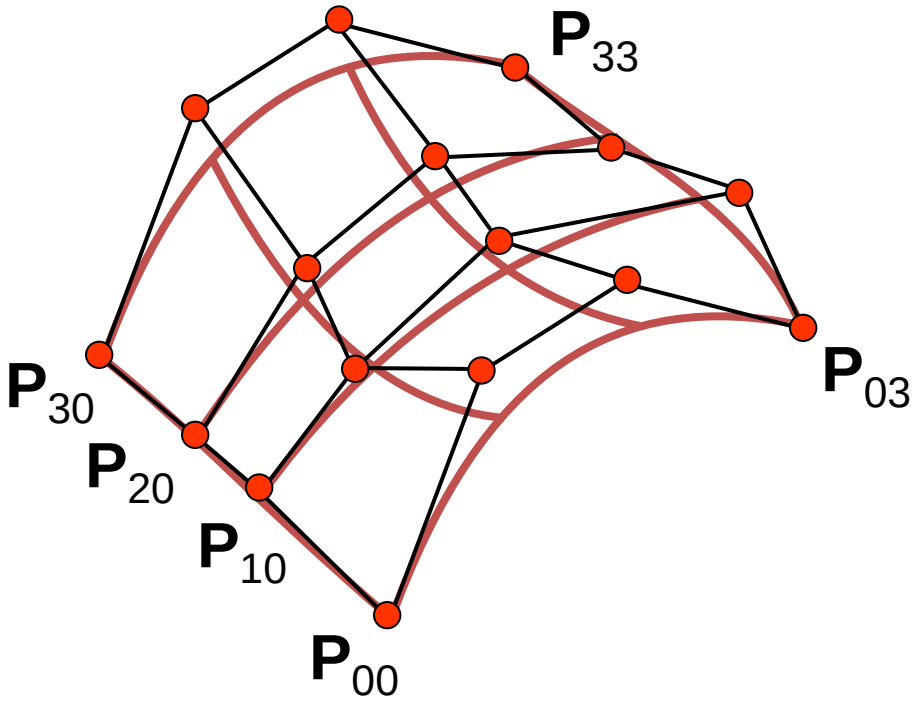
With τ being the "tension" or sharpness at the control points (e.g., 0.5)



Spline Surface

Control points : 4×4 matrix of points $\mathbf{P}_{ij}$

Define a smooth patch.

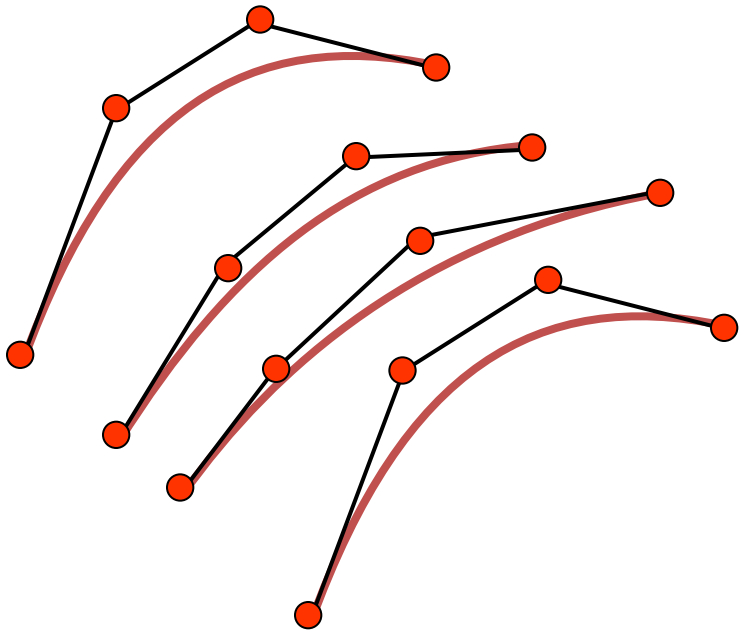


Spline Surface

Control points : 4×4 matrix of points $\mathbf{P}_{ij}$

Define four curves.

Surface : Smooth these four curves.

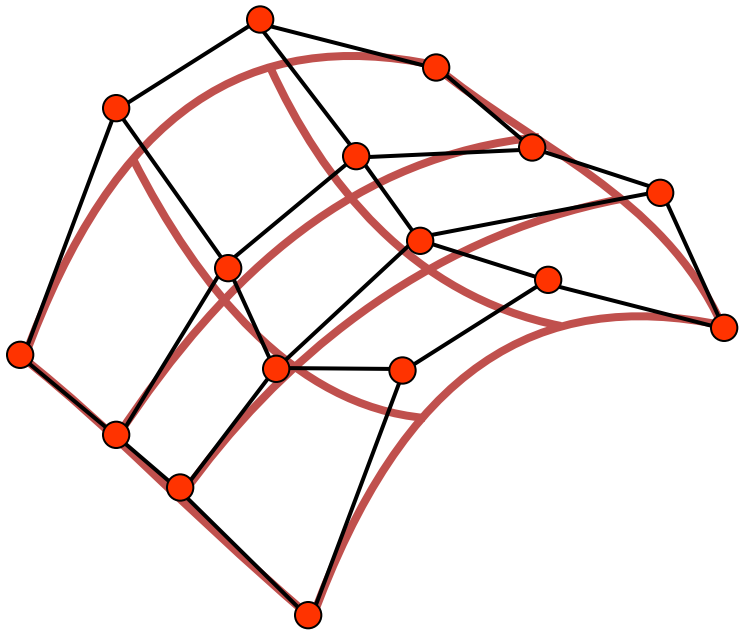


Spline Surface

Control points : 4×4 matrix of points $\mathbf{P}_{ij}$

Define four curves.

Surface : Smooth these four curves.



Spline Surface

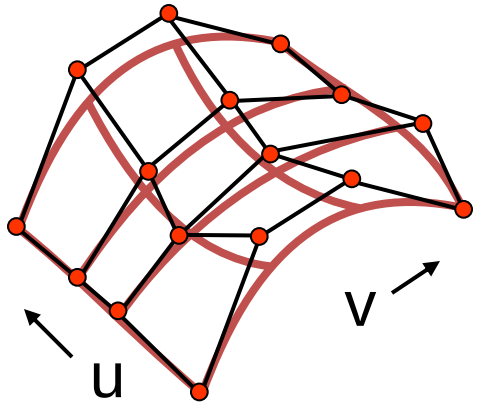
Surface :

Control points : 4×4 matrix of points $\mathbf{P}_{i,j}$

$$\text{Curve} : \mathbf{P}(u) = \sum_{k=0}^{3} B_k(u)\mathbf{P}_k$$

$$\text{Surface} : \mathbf{P}(u,v) = \sum_{i=0}^{3} B_i(u)\left(\sum_{j=0}^{3} B_j(v)\mathbf{P}_{ij}\right)$$

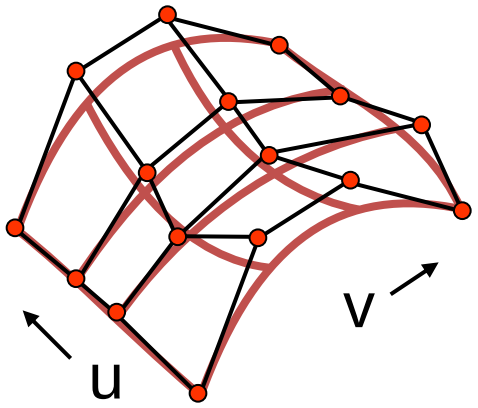
$$= \sum_{i=0}^{3}\sum_{j=0}^{3} B_i(u)B_j(v)\mathbf{P}_{ij}$$



Spline Surface

Display of surface

$$\mathbf{P}(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} B_i(u)B_j(v)\mathbf{P}_{ij}$$



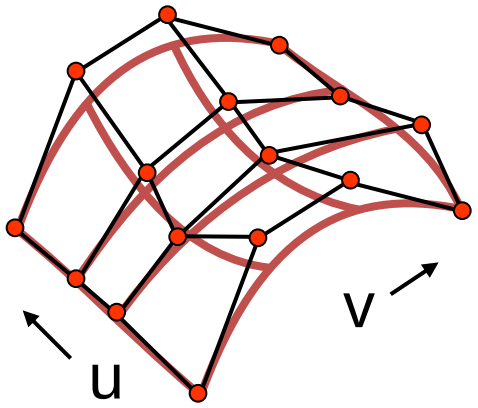

```
du := 1/nu;  // nu: #facets u-direction
dv := 1/nv;  // nv: #facets v-direction
for i := 0 to nu–1 do
    u := i*du;
    for j := 0 to nv – 1 do
        v := j*dv;
        DrawQuad(P(u,v), P(u+du, v), P(u+du, v+dv), P(u, v+dv))
```

Spline Surface

```
// Alternative: calculate points first

for i := 0 to nu do
for j := 0 to nv do
        Q[i, j] := P(i/nu, j/nv);

for i := 0 to nu – 1 do
for j := 0 to nv – 1 do
   DrawQuad(Q[i, j], Q[i+1, j], Q[i+1, j+1], Q[i, j+1])
```

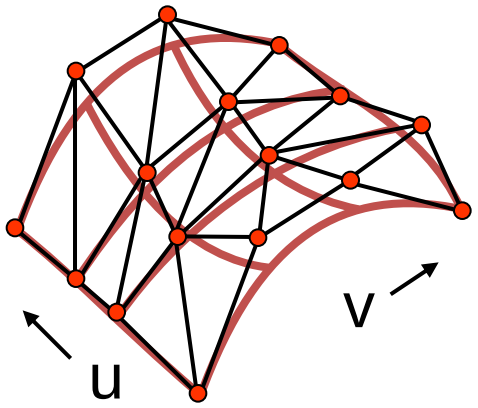


Spline Surface

```
// Alternative: calculate points first,
// triangle version

for i := 0 to nu do
for j := 0 to nv do
        Q[i, j] := P(i/nu, j/nv);

for i := 0 to nu – 1 do
for j := 0 to nv – 1 do
   DrawTriangle(Q[i, j], Q[i+1, j], Q[i+1, j+1]);
   DrawTriangle(Q[i, j], Q[i+1, j+1], Q[i, j+1]);
```

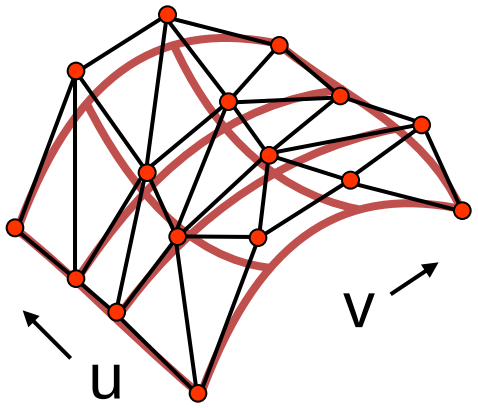


Spline Surface

```
// Alternative: calculate points first,
// triangle variant, triangle strip

for i := 0 to nu do
for j := 0 to nv do
        Q[i, j] := P(i/nu, j/nv);

for i := 0 to nu – 1 do
   glBegin(GL_TRIANGLE_STRIP);
   for j := 0 to nv – 1 do
       glVertex(Q[i, j]);
       glVertex(Q[i, j+1]);
   glEnd;
```

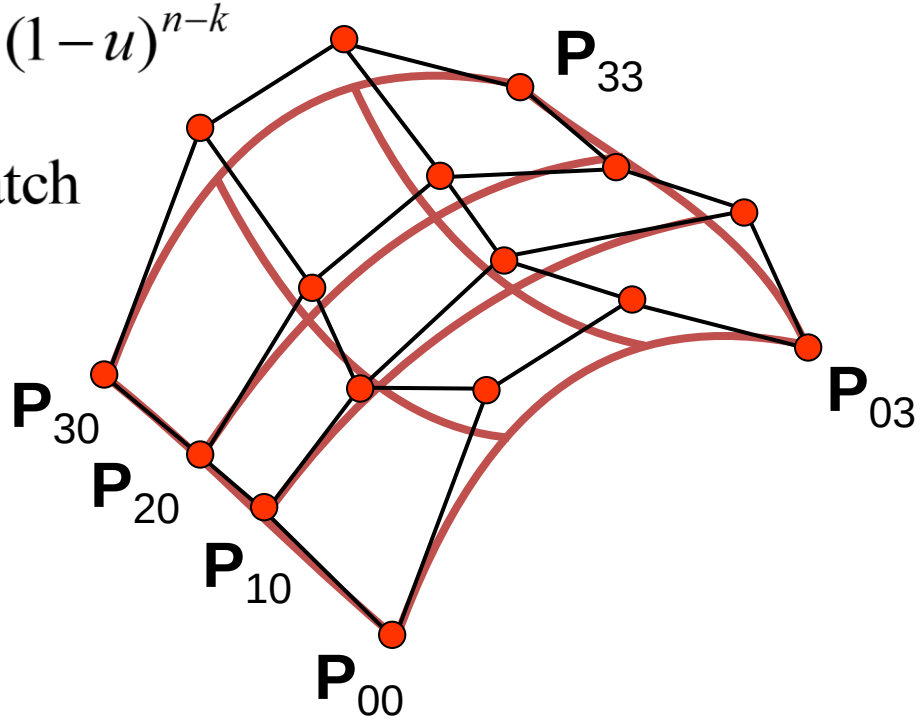


Bézier Surface Patches

$$\text{Generic}: \mathbf{P}(u,v) = \sum_{j=0}^{m}\sum_{k=0}^{n} B_j(v) B_k(u) \mathbf{P}_{j,k}$$

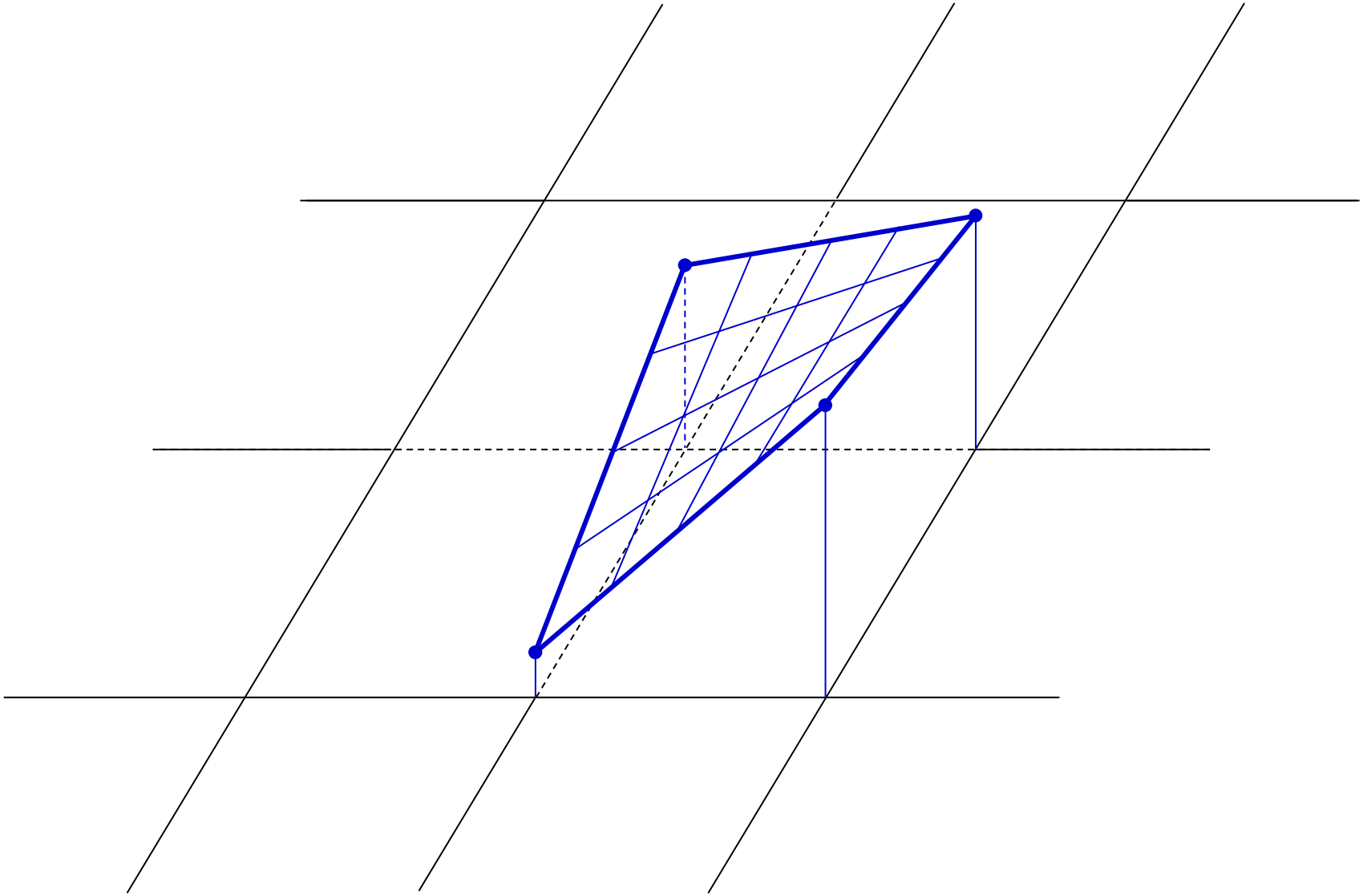
$$\text{with } B_{k,n}(u) = \frac{n!}{k!(n-k)!} u^k (1-u)^{n-k}$$

$$\text{Often } m = n = 4 : \text{bicubic patch}$$



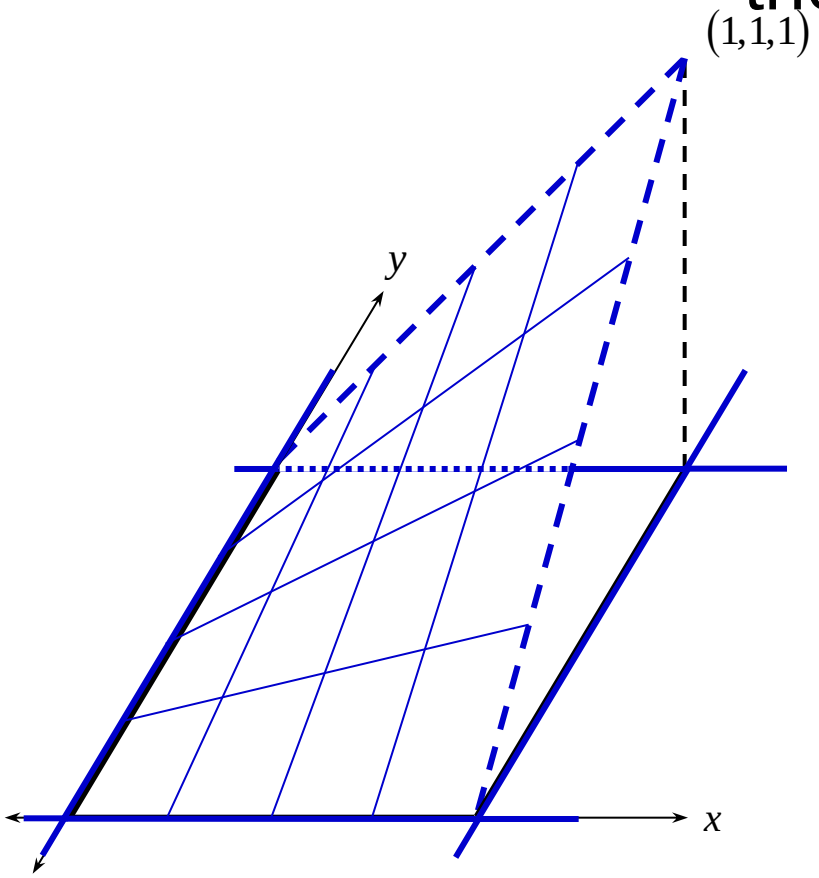
Lets get a bit more complicated…

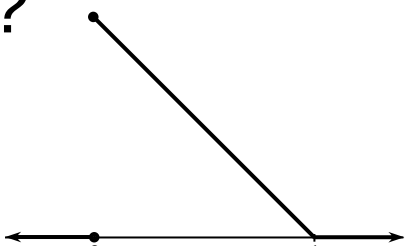
A closer view of four data points



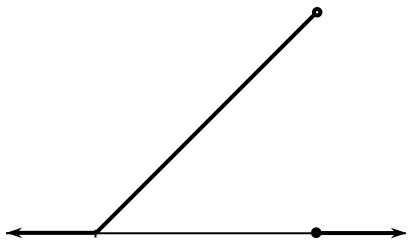
There are two 1-D elementary basis functions.

How many 2-D elementary basis functions are there?





$\theta_1(t)$



$\theta_2(t)$

We define the bilinear 2-D elementary basis functions as follows:

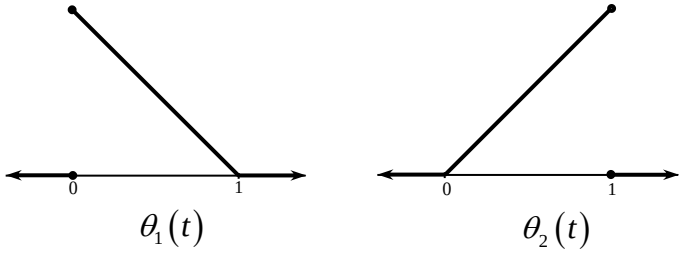
$$\theta_{11}(x,y) = \theta_1(x)\theta_1(y)$$

$$\theta_{12}(x,y) = \theta_1(x)\theta_2(y)$$

$$\theta_{21}(x,y) = \theta_2(x)\theta_1(y)$$

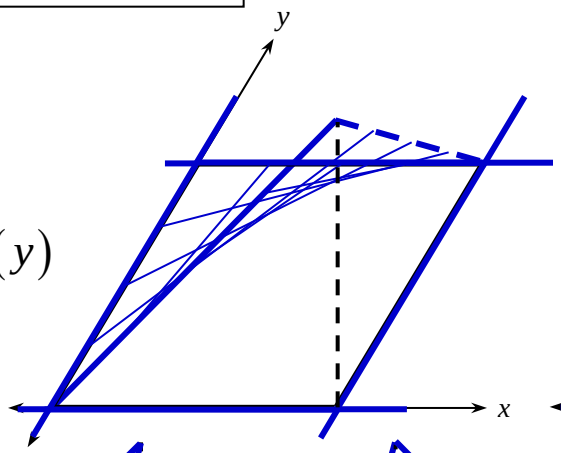
$$\theta_{22}(x,y) = \theta_2(x)\theta_2(y)$$

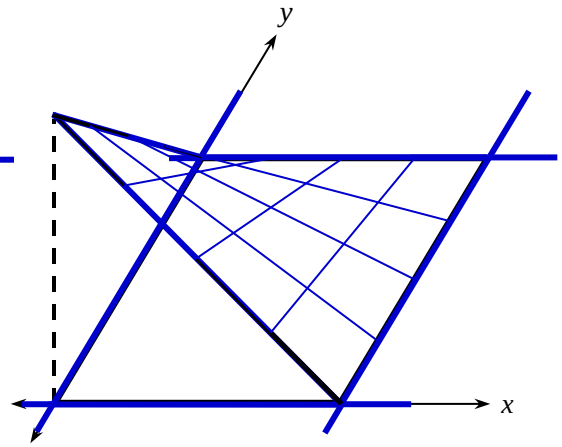
The two 1-D elementary basis functions



The four bilinear 2-D elementary basis functions

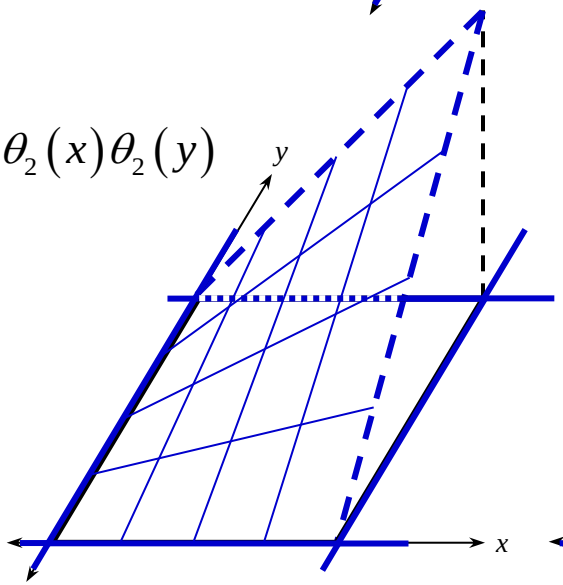
$$\theta_{21}(x,y) = \theta_2(x)\theta_1(y)$$

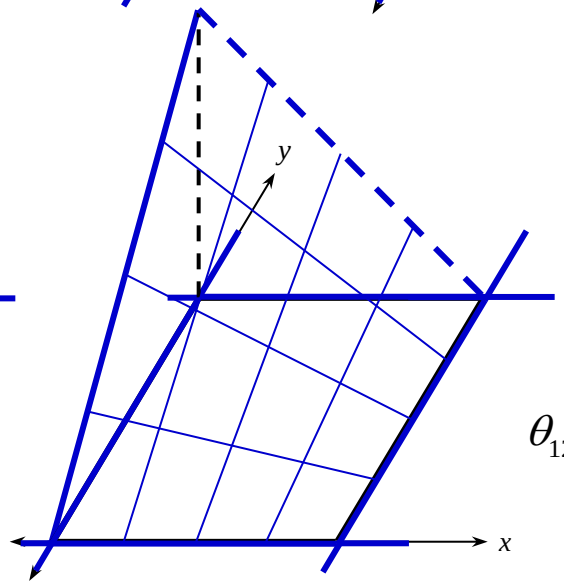




$$\theta_{11}(x,y) = \theta_1(x)\theta_1(y)$$

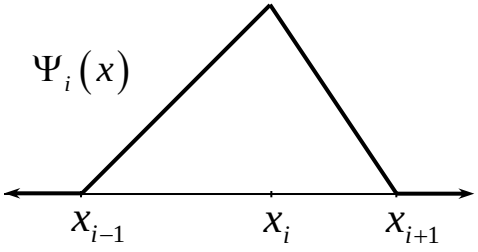
$$\theta_{22}(x,y) = \theta_2(x)\theta_2(y)$$





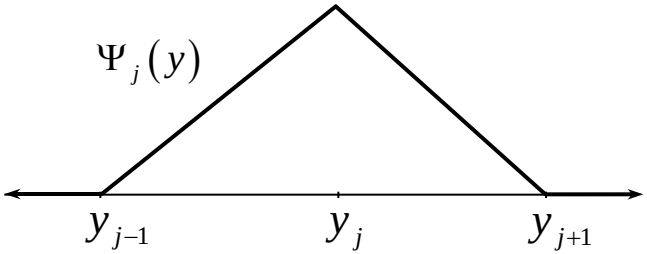
$$\theta_{12}(x,y) = \theta_1(x)\theta_2(y)$$

We can construct the 2-D bilinear spline basis functions directly from the 1-D linear spline basis functions

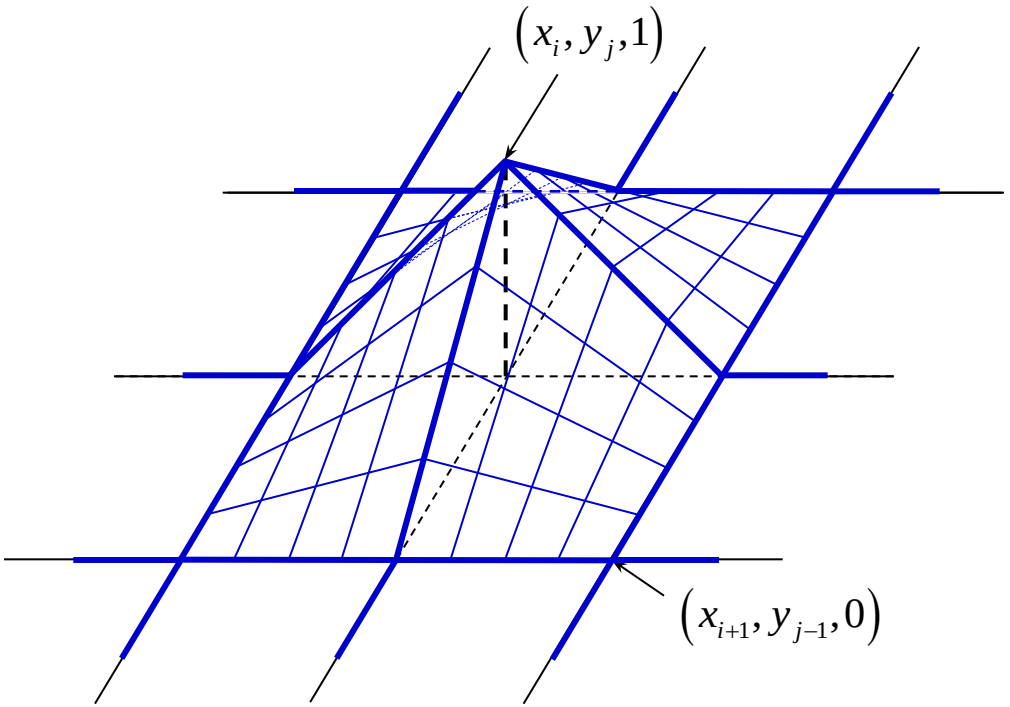


We define the ijth 2-D bilinear spline basis function as follows:

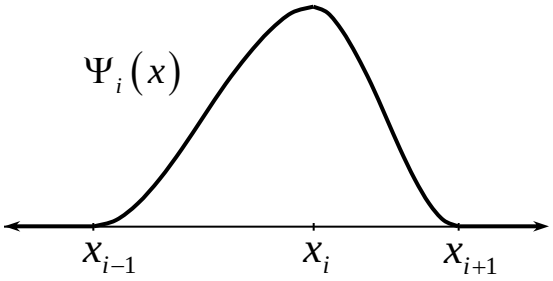
$$\Psi_{ij}(x,y) = \Psi_i(x)\Psi_j(y)$$

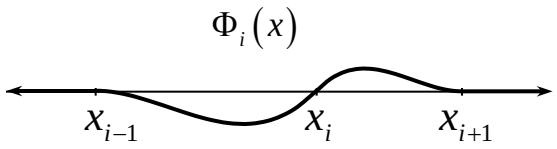


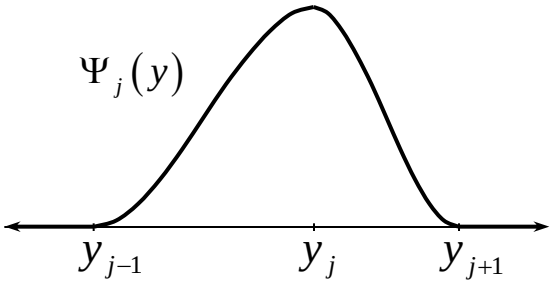
As in the 1-D case we want our basis functions to be zero at all but one of the data points.

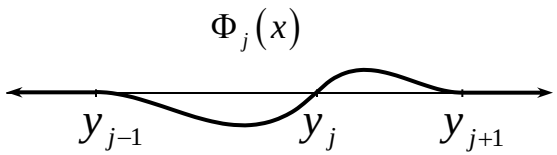


We can construct the 2-D bicubic spline basis functions directly from the 1-D bicubic spline basis functions









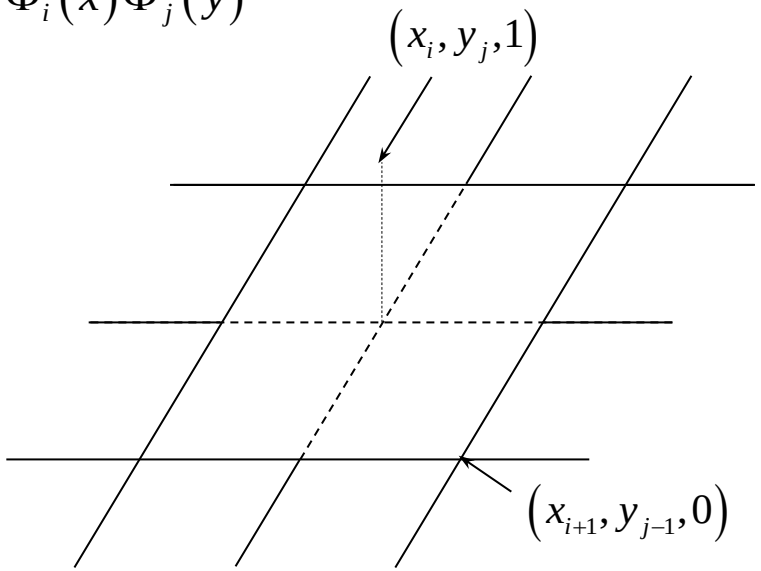
At each data point we define <u>four</u> 2-D bicubic spline basis functions as follows:

$$\Theta^{\Psi\Psi}{}_{ij}(x,y) = \Psi_i(x)\Psi_j(y)$$

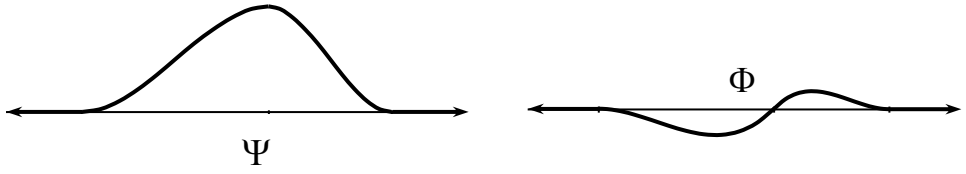
$$\Theta^{\Psi\Phi}{}_{ij}(x,y) = \Psi_i(x)\Phi_j(y)$$

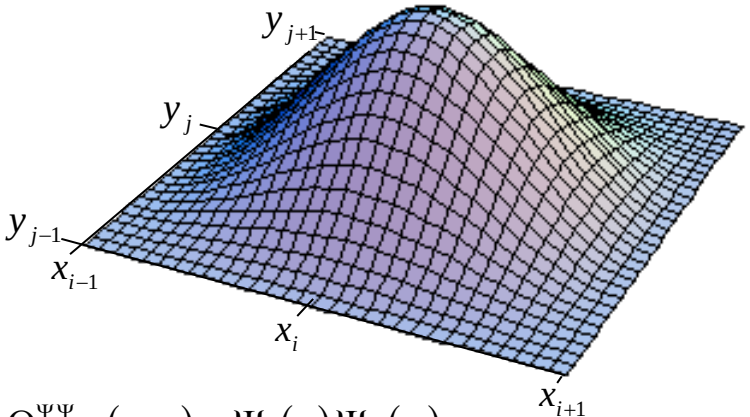
$$\Theta^{\Phi\Psi}{}_{ij}(x,y) = \Phi_i(x)\Psi_j(y)$$

$$\Theta^{\Phi\Phi}{}_{ij}(x,y) = \Phi_i(x)\Phi_j(y)$$

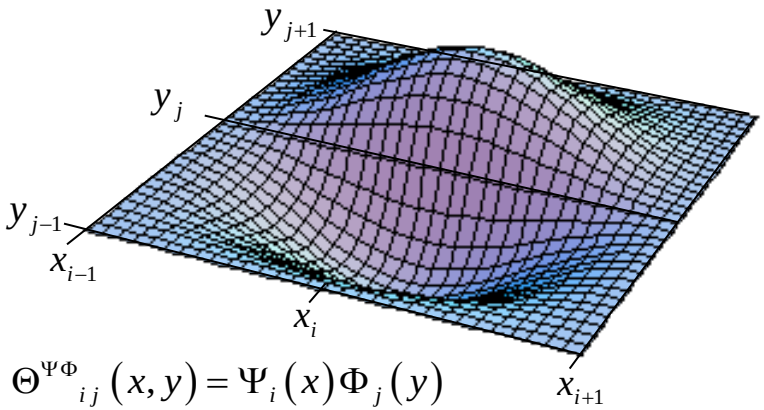


The 2-D bicubic spline basis functions

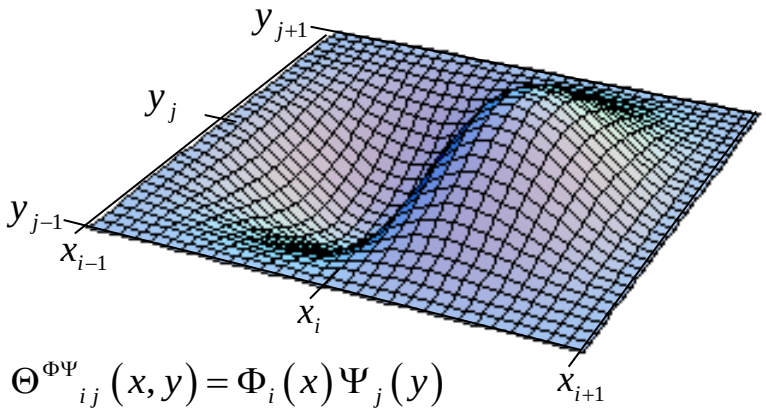




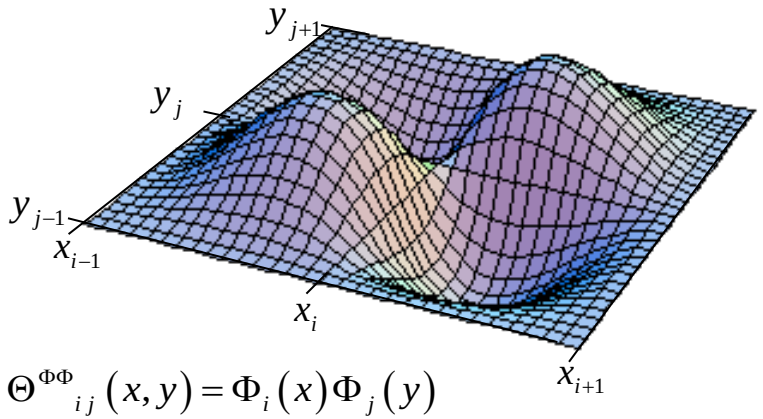
$$\Theta^{\Psi\Psi}{}_{ij}(x,y) = \Psi_i(x)\Psi_j(y)$$



$$\Theta^{\Psi\Phi}{}_{ij}(x,y) = \Psi_i(x)\Phi_j(y)$$



$$\Theta^{\Phi\Psi}{}_{ij}(x,y) = \Phi_i(x)\Psi_j(y)$$



$$\Theta^{\Phi\Phi}{}_{ij}(x,y) = \Phi_i(x)\Phi_j(y)$$

What can we do?

- With 2D elementary basis functions?
- With 2D bilinear spline basis?
- With 2D bicubic basis functions?

Welcome to the world of Basis functions!

- Radial Basis Functions
- Haar Basis (Wavelets):