Linear Algebra for Computer Graphics

CS535

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Linear Algebra

- Why do we need it?
  - Geometric operations
    - Solving “mini geometric problems”…
  - Modeling transformation
    - Move “objects” into place relative to a world origin
  - Viewing transformation
    - Move “objects” into place relative to camera
  - Perspective transformation
    - Project “objects” onto image plane

Is this new stuff?

- No its not…but that does not mean its not critical….
- Early Roots (Ancient Times)
  - Babylonians (c. 1800 BC): Solved simple 2x2 linear equations.
  - Chinese (c. 200 AD): Published methods for solving 3x3 systems in ”The Nine Chapters on the Mathematical Art”.

Is this new stuff?

- Emergence of Modern Concepts (17th-18th Century)
  - 1637 (René Descartes): Introduced coordinate geometry, linking lines/planes to linear equations.
  - 1693 (Gottfried Leibniz): Developed determinants for solving systems, though forgotten until Cramer.
  - 1750 (Gabriel Cramer): Used determinants to solve systems (Cramer's Rule).

Is this new stuff?

- Formalization & Expansion (19th Century)
  - 1843 (William Rowan Hamilton): Developed quaternions, extending complex numbers.
  - 1844 (Hermann Grassmann): Published The Extension Theory, laying the foundation for modern linear algebra.
  - 1848 (James Joseph Sylvester): Introduced the term "matrix".
  - 1858 (Arthur Cayley): Defined matrix multiplication and inverses, formalizing matrix algebra.

Is this new stuff?

- Since so old…

  you should know this stuff inside-out!

“New” Books

- Handbook of Mathematics and Computational Science (1998)
  - Harris and Stocker

    “everything is there…”

- 3D Game Engine Design (2006)
  - Eberly

    “more than just games…”

Points, Vectors, and more

- A point $\mathbf{p} = (p_x, p_y, p_z)$ defines a location
- A vector $\mathbf{v} = (v_x, v_y, v_z)$ defines a direction
- A ray is a point with a direction: $\boldsymbol{r}(t) = \boldsymbol{p} + \boldsymbol{v}t$
- A line is an infinite line (e.g., $y = mx + b$)
- A line segment is a piece of line, say between a point a and a point b (e.g., $\boldsymbol{l}(t) = \boldsymbol{a}(1 - t) + \boldsymbol{b}t$

Basic geometric computations

- Distance of point to line? (2D or 3D)
- Is a point in a plane? (3D)
- Is a point inside a triangle? (2D)
- What is distance between to lines (3D)?

Transformations

- Most popular transformations in graphics
  - Translation
  - Rotation
  - Scale
  - Projection
- In order to use a single matrix for all, we use homogeneous coordinates…

Homogenous Coordinates

- A 4x4 matrix embeds a rotation, translation, as well as other matrix transformations including projection

- A vector is $[v_x \; v_y \; v_z \; 0]^T$

- A point is $[p_x \; p_y \; p_z \; 1]^T$

3D Transformations

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Identity

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & tx \\ 0 & 1 & 0 & ty \\ 0 & 0 & 1 & tz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Mirror over X axis

3D Transformations

Rotate around Z axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & 0 \\ \sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around Y axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\Theta & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

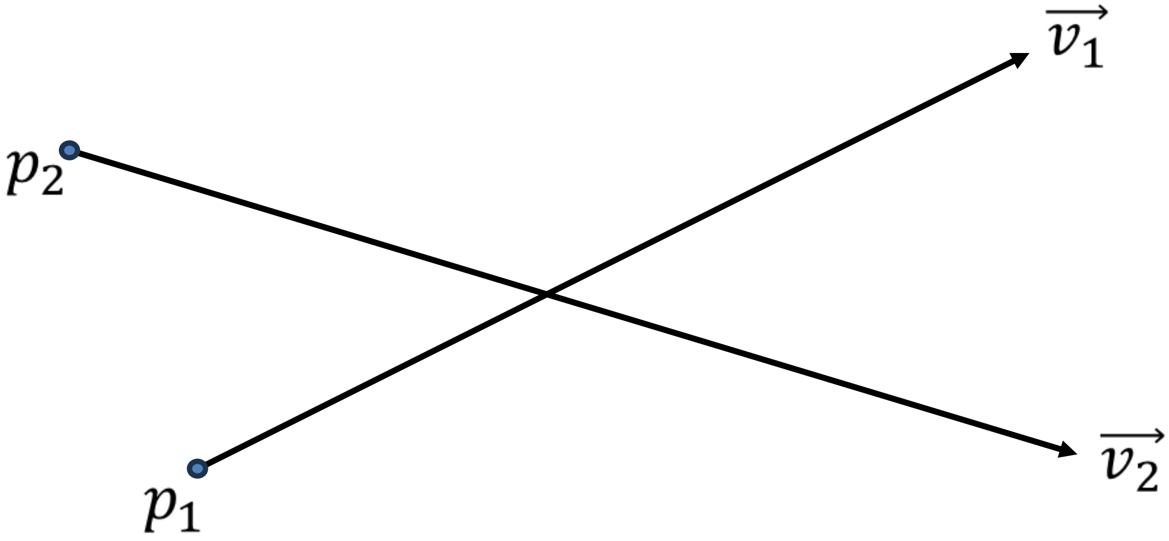
and many more…

Rotate around X axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta & 0 \\ 0 & \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

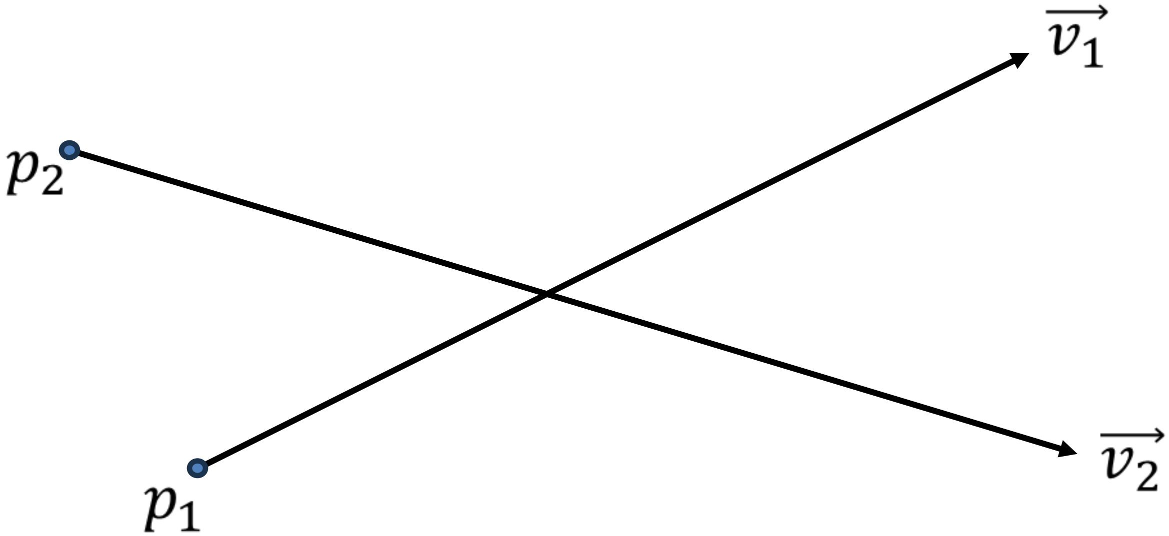
Let’s Play

- When do these 2 non-parallel lines (in 3D) intersect?



Let's Play

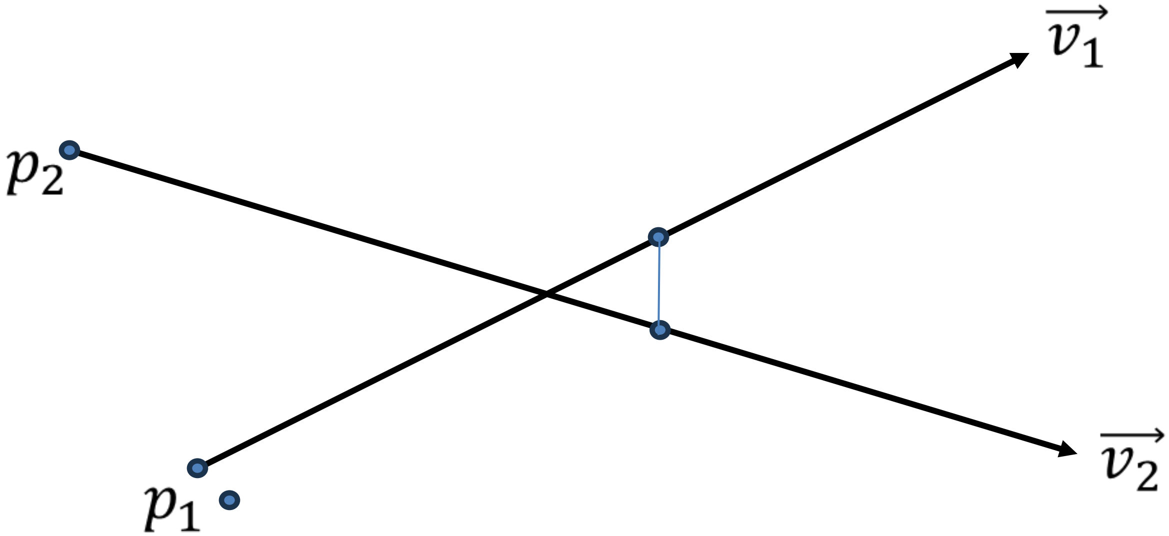
- When do these 2 non-parallel lines (in 3D) intersect?



e.g., $(\overrightarrow{v_1}\ x\ \overrightarrow{v_2}) \cdot p_1 p_2 = 0$

Let's Play

- What is the distance between these 2 lines?



Let’s Play

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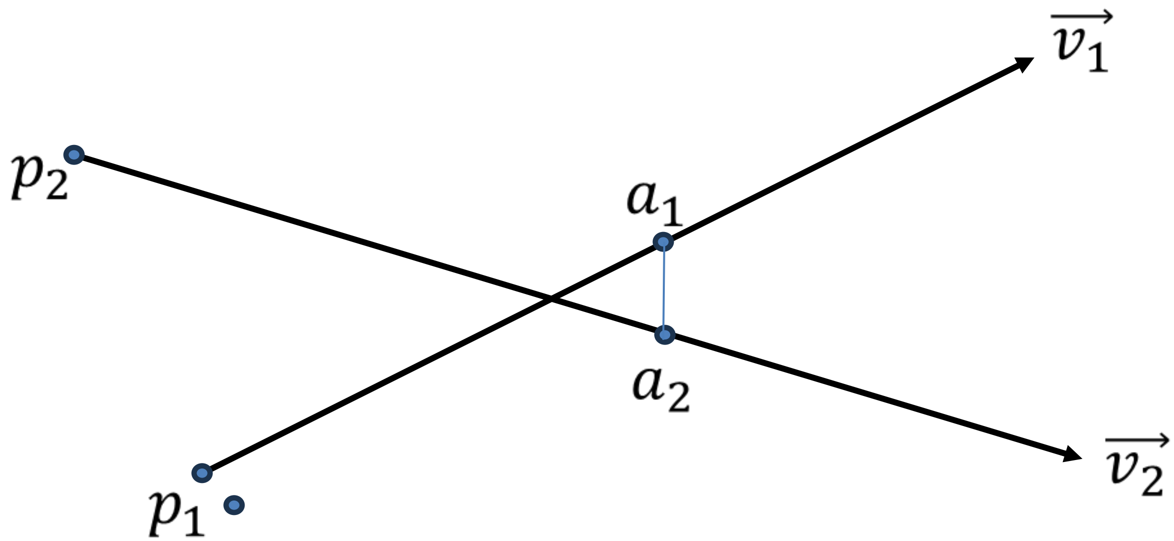
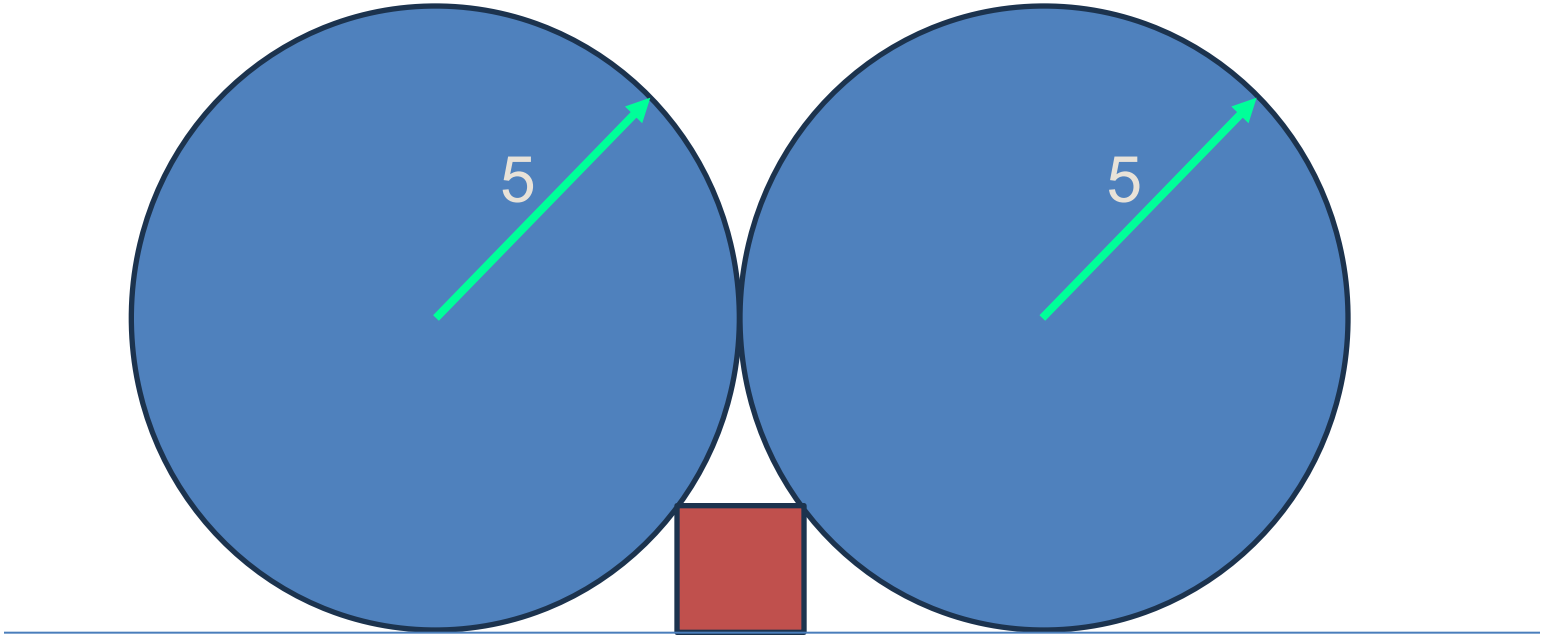


$$d = |(v_1 x v_2) \cdot (a_2 - a_1)| / ||v_1 x v_2||$$

Let's Play

- What is area of the red square?



Let's Play

- What is area of the red square?
  - Let side of square be x
  - Add a radial line from circle center to corner of square – it has length 5
  - Its vertical edge has length $(5 - x)$
  - Its horizontal edge has length $(5 - x/2)$
  - Thus $(5 - x)^2 + \left(5 - \frac{x}{2}\right)^2 = 5^2$
  - Thus $(x - 2)(x - 10) = 0$