Information Contraction and Decomposition

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Doctoral Thesis Defense
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Thesis Committee

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Outline

Preliminaries

- finite alphabets $\mathcal{X}$ and $\mathcal{Y}$

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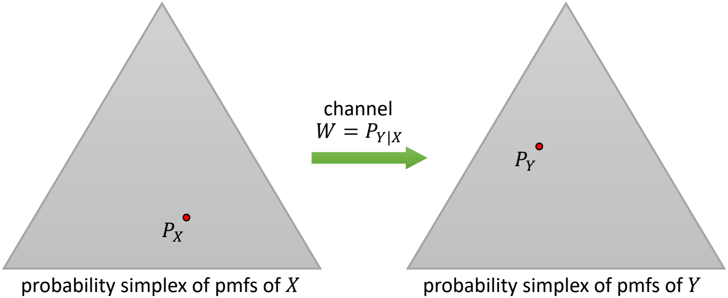
- probability distributions are *row* vectors
  e.g. P_X is pmf on $\mathcal{X}$, and P_Y is pmf on $\mathcal{Y}$
- channels (conditional distributions) are *row stochastic* matrices
  e.g. $W = P_{Y|X}$ such that $P_Y = P_X W$



f-Divergence

Definition (f-Divergence [Csi63, Mor63, AS66, ZZ73, Aka73])

For any convex function $f : (0, \infty) \to \mathbb{R}$ such that $f(1) = 0$, we define the f-divergence between any two pmfs R_X and P_X on $\mathcal{X}$ as:

$$D_f(R_X || P_X) \triangleq \sum_{x \in \mathcal{X}} P_X(x)\, f\left(\frac{R_X(x)}{P_X(x)}\right)$$

where $f(0) = \lim_{t \to 0} f(t)$, $0\, f\left(\frac{0}{0}\right) = 0$, and $0\, f\left(\frac{r}{0}\right) = \lim_{p \to 0} p\, f\left(\frac{r}{p}\right)$ for all $r > 0$.

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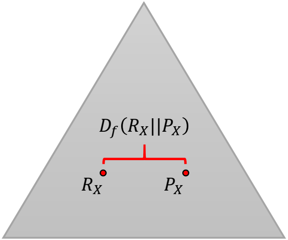
  "Distance" between distributions



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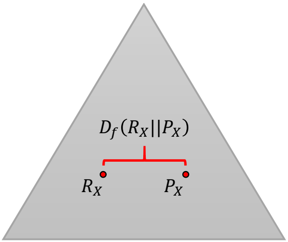
- **Non-negativity:**

$$D_f(R_X||P_X) \geq 0$$

with equality iff $R_X = P_X$ (where we assume that f is strictly convex at 1)



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- **Total Variation (TV) Distance:** $f(t) = \frac{1}{2}|t-1|$

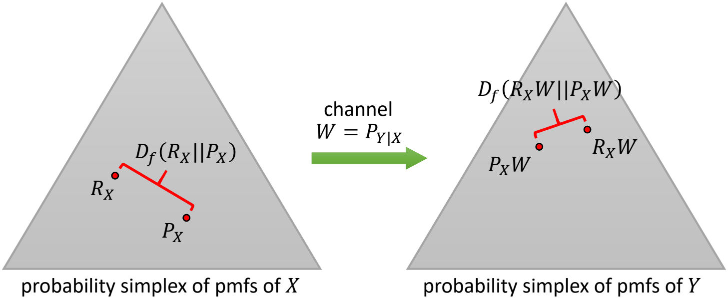
$$\|R_X - P_X\|_{\mathsf{TV}} = \frac{1}{2} \sum_{x \in \mathcal{X}} |R_X(x) - P_X(x)|$$

Data Processing Inequality (DPI)

Prop (DPI for f-Divergences [Csi63, Mor63, AS66, ZZ73])

Given channel $W = P_{Y|X}$, for any two pmfs R_X and P_X on $\mathcal{X}$:

$$D_f(R_X W || P_X W) \leq D_f(R_X || P_X) \, .$$



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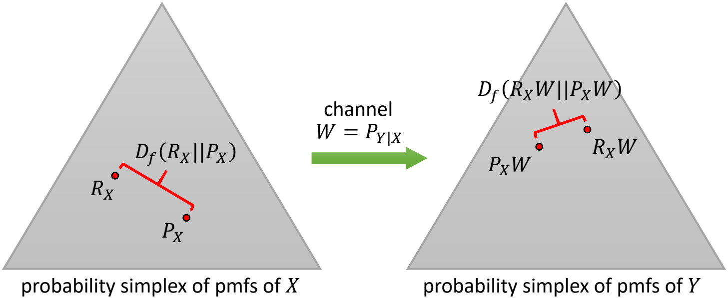

Intuition: R_X and P_X are "less distinguishable" from noisy observation Y compared to true data X.

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Want *stronger* version of DPI:

$$D_f(R_X W^n || P_X) \leq \eta^n D_f(R_X || P_X)$$

for some coefficient $\eta \in (0, 1)$.

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For a fixed channel $W = P_{Y|X}$, the contraction coefficient for an f-divergence is:

$$\eta_f(P_{Y|X}) \triangleq \sup_{\substack{R_X, P_X: \\ 0 < D_f(R_X || P_X) < +\infty}} \frac{D_f(R_X W || P_X W)}{D_f(R_X || P_X)} \, .$$

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- Properties of contraction coefficients I well-studied [CIRRSZ93].

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Properties: Contraction Coefficients of Sources & Channels

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- Is there an upper bound on η_f in terms of η_{χ^2}?

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Example: This holds for *Hellinger divergences* of order $\alpha \in (0, 2]\backslash\{1\}$, i.e. $f(t) = \frac{t^\alpha - 1}{\alpha - 1}$.

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Corollary (KL Contraction Coefficient Bound)

$$\eta_{\mathsf{KL}}(P_X, P_{Y|X}) \leq \frac{\eta_{\chi^2}(P_X, P_{Y|X})}{p_\star}$$

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Theorem (Refined KL Contraction Coefficient Bound)

$$\eta_{\mathsf{KL}}(P_X, P_{Y|X}) \leq \frac{2\,\eta_{\chi^2}(P_X, P_{Y|X})}{\phi\left(\max_{A\subseteq\mathcal{X}} \min\{P_X(A), P_X(A^c)\}\right) p_\star} \leq \frac{\eta_{\chi^2}(P_X, P_{Y|X})}{p_\star}$$

where $\phi(p) = \frac{1}{1-2p}\log\left(\frac{1-p}{p}\right)$.

Proof Idea: Use bounds between f-divergences and χ^2-divergence based on [Su95, OW05, Gil10, Rag16].

Illustration of KL Contraction Coefficient Bounds

Suppose $X, Y \in \{0, 1\}$ such that $X \sim \text{Ber}(\mathbb{P}(X = 1))$ and $P_{Y|X}$ is *binary symmetric channel* (BSC) with crossover probability $p \in [0, 1]$.

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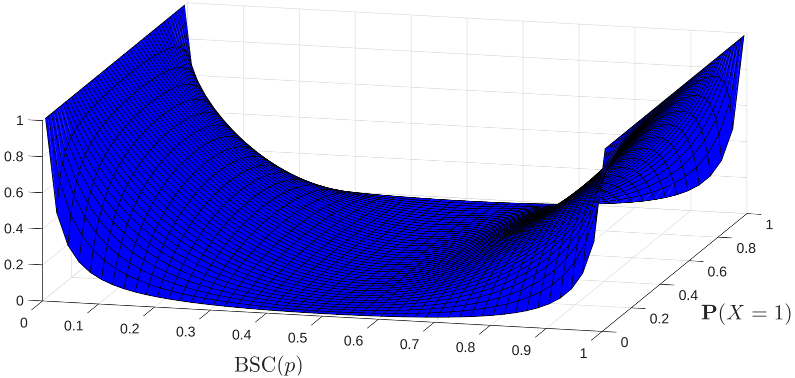

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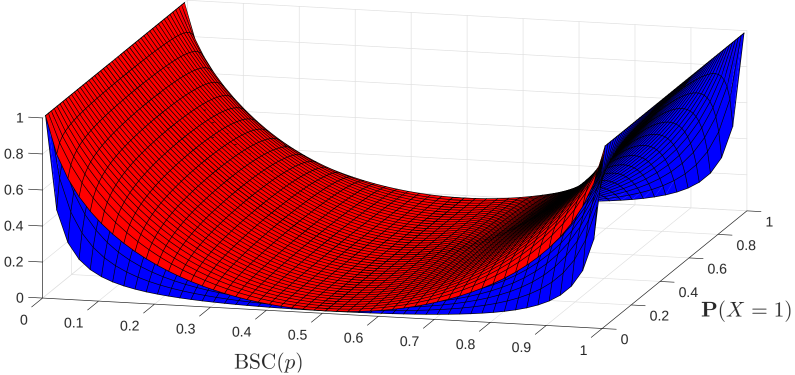
$$\eta_{\chi^2}(P_X, P_{Y|X}) \leq \eta_{\text{KL}}(P_X, P_{Y|X})$$



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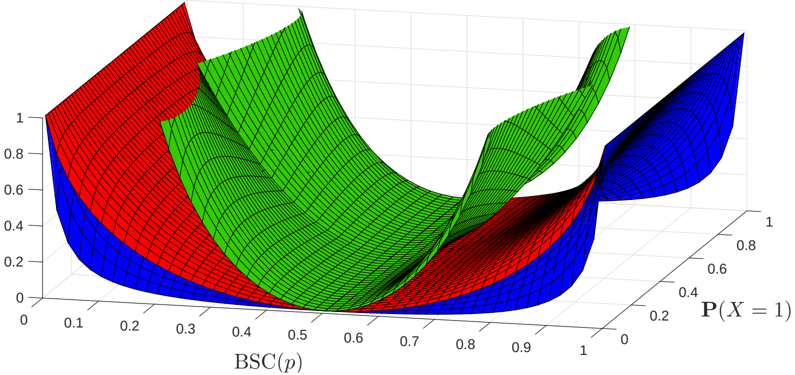

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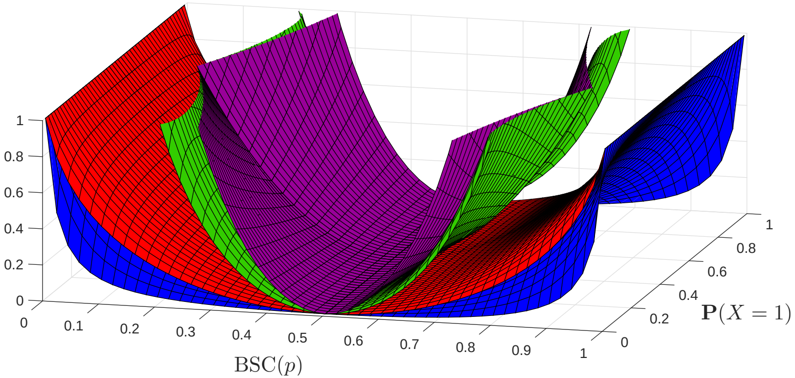


Outline

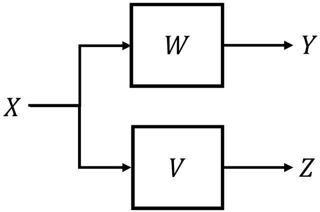
Less Noisy Preorder over Channels

Definition (Less Noisy Preorder [KM77])

$P_{Y|X} = W$ is less noisy than $P_{Z|X} = V$, denoted $W \succeq_{\text{ln}} V$, if and only if:

$$D(P_X W || Q_X W) \geq D(P_X V || Q_X V)$$

for every pair of input distributions P_X and Q_X.



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- Why $\succeq_{\text{ln}}$ domination by symmetric channels?
  - extend SDPIs because we ♥ information theory
  - $\succeq_{\text{ln}}$ domination $\Rightarrow$ log-Sobolev inequality

Motivation: Extend SDPI

SDPI for KL divergence [AG76]:

For any channel V, for all pairs of pmfs P_X, Q_X:

$$\eta_{\mathsf{KL}}(V)\, D(P_X||Q_X) \geq D(P_X V||Q_X V)$$

where $\eta_{\mathsf{KL}}(V) \in [0,1]$ is the contraction coefficient.

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- **Definition:** q-ary erasure channel $q\text{-}EC(1-\eta)$ erases input w.p. $1-\eta$, and reproduces input w.p. η.

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SDPI $\Leftrightarrow$ $\succeq_{\mathsf{ln}}$ domination by erasure channel

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Channel matrix:

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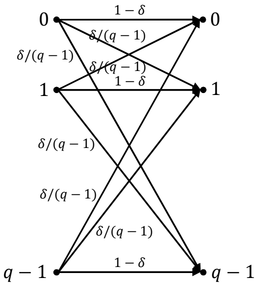

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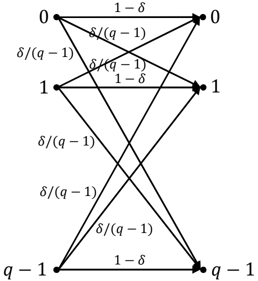

Remark: For every channel V, $W_0 \succeq_{\mathsf{ln}} V$ and $V \succeq_{\mathsf{ln}} W_{(q-1)/q}$.

Outline

Operator Convexity

$f : \mathbb{R} \to \mathbb{R}$ can be applied to an $n \times n$ Hermitian matrix A via:

$$f(A) = U \operatorname{diag}(f(\lambda_1), \ldots, f(\lambda_n)) U^H$$

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where $\succeq_{\text{PSD}}$ is the Löwner partial order.

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Löwner-Heinz Theorem (Examples [Löw34, Hei51])

- For every $\alpha \in (0, 2] \backslash \{1\}$, $f : (0, \infty) \to \mathbb{R}$, $f(t) = \frac{t^\alpha - 1}{\alpha - 1}$ is operator convex.
- $f : (0, \infty) \to \mathbb{R}$, $f(t) = t \log(t)$ is operator convex.

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Characterization of Less Noisy using Operator Convexity

Theorem (Equivalent Characterizations of $\succeq_{\text{ln}}$)

Given channels W and V, and any non-linear operator convex function $f : (0, \infty) \to \mathbb{R}$ such that $f(1) = 0$:

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- Let $J_X = P_X - Q_X$. Then, we have:

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Condition for Degradation by Symmetric Channels

Given channel V, find q-ary symmetric channel W_δ with largest $\delta \in \left[0, \frac{q-1}{q}\right]$ such that $W_\delta \succeq_{\mathsf{ln}} V$?

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Theorem (Degradation by Symmetric Channels)

For channel V with common input and output alphabet, and minimum probability entry $\nu = \min\{[V]_{i,j} : 1 \leq i,j \leq q\}$:

$$0 \leq \delta \leq \frac{\nu}{1-(q-1)\nu + \frac{\nu}{q-1}} \ \Rightarrow \ W_\delta \succeq_{\text{deg}} V \, .$$

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Remark: Condition is tight when no further information about V known. For example, suppose:

$$V = \begin{bmatrix} \nu & 1-(q-1)\nu & \nu & \cdots & \nu \\ 1-(q-1)\nu & \nu & \nu & \cdots & \nu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-(q-1)\nu & \nu & \nu & \cdots & \nu \end{bmatrix} .$$

Then, $0 \leq \delta \leq \nu / \left(1-(q-1)\nu + \frac{\nu}{q-1}\right) \;\Leftrightarrow\; W_\delta \succeq_{\text{deg}} V$.

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- q-ary symmetric channel: $P_Z = \left(1 - \delta, \frac{\delta}{q-1}, \ldots, \frac{\delta}{q-1}\right)$ for $\delta \in [0, 1]$

$$(\cdot * P_Z) = W_\delta$$

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Domination Structure of Additive Noise Channels

Theorem (More Noisy and Degradation Regions)

For W_δ with $\delta \in \left[0, \frac{q-1}{q}\right]$ and $q \geq 2$:

$$\begin{aligned} \textit{degrade}(W_\delta) &= \textit{conv}(\text{rows of } W_\delta) \\ &\subseteq \textit{conv}(\text{rows of } W_\delta \text{ and } W_\gamma) \\ &\subseteq \textit{more-noisy}(W_\delta) \\ &\subseteq \{P_Z : \|P_Z - \mathbf{u}\|_2 \leq \|w_\delta - \mathbf{u}\|_2\} \end{aligned}$$

where $\textit{conv}(\cdot)$ denotes convex hull, $\gamma = (1-\delta)/\left(1-\delta+\frac{\delta}{(q-1)^2}\right)$, $\mathbf{u}$ is the uniform pmf, and w_δ is first row of W_δ.

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Furthermore, $\textit{more-noisy}(W_\delta)$ is closed, convex, and invariant under permutations corresponding to $(\mathcal{X}, \oplus)$.

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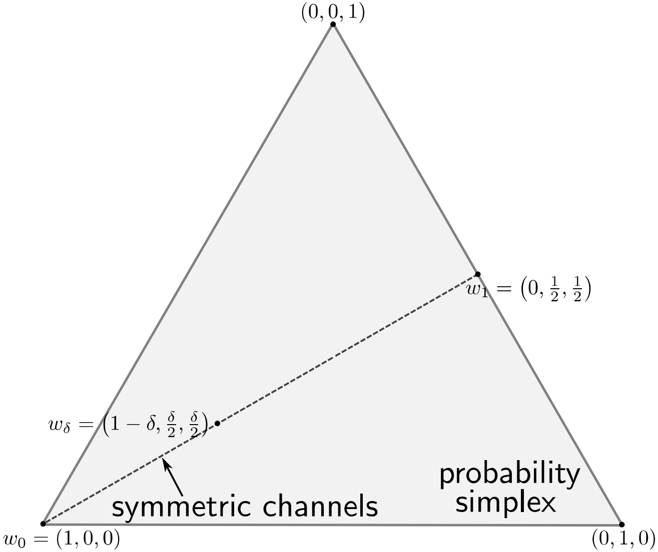
Illustration of the $q = 3$ case:



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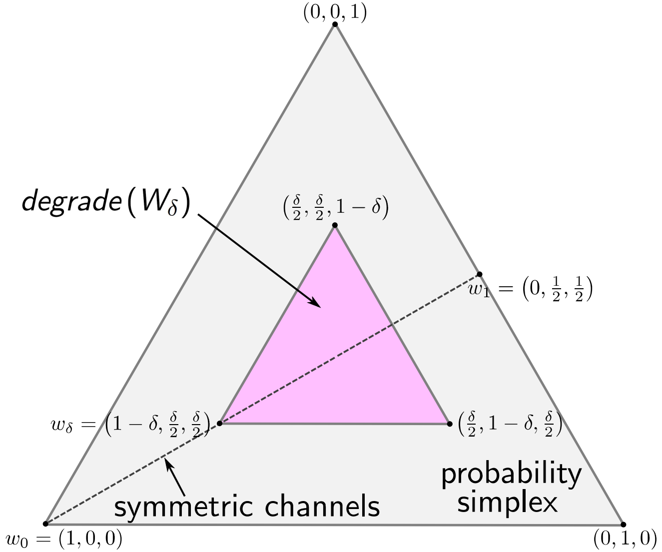

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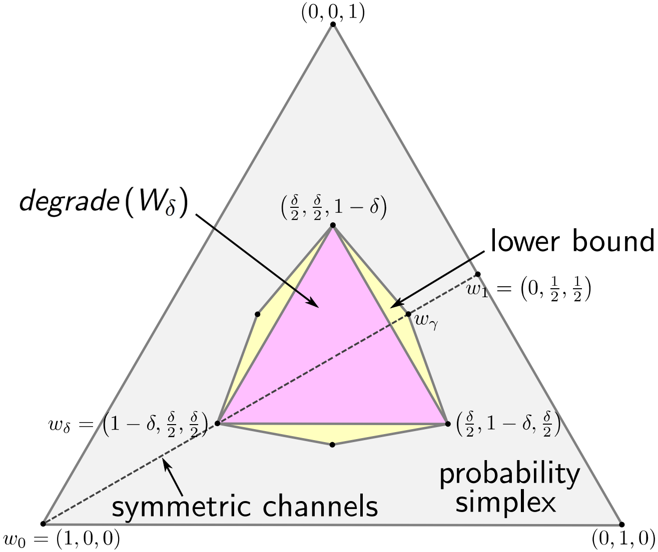

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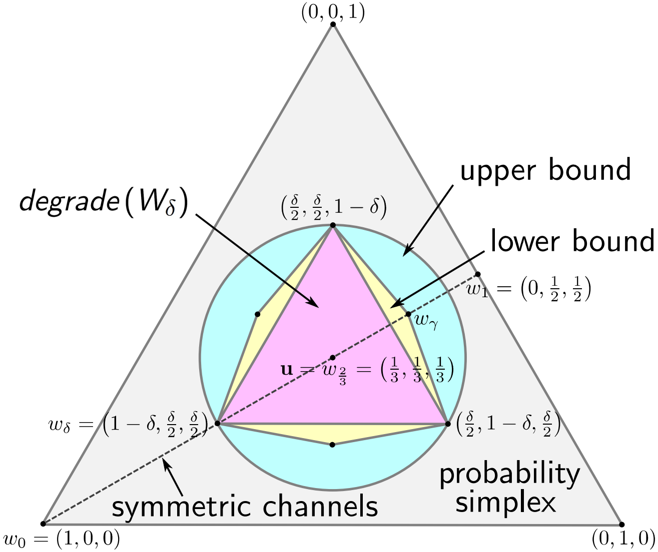

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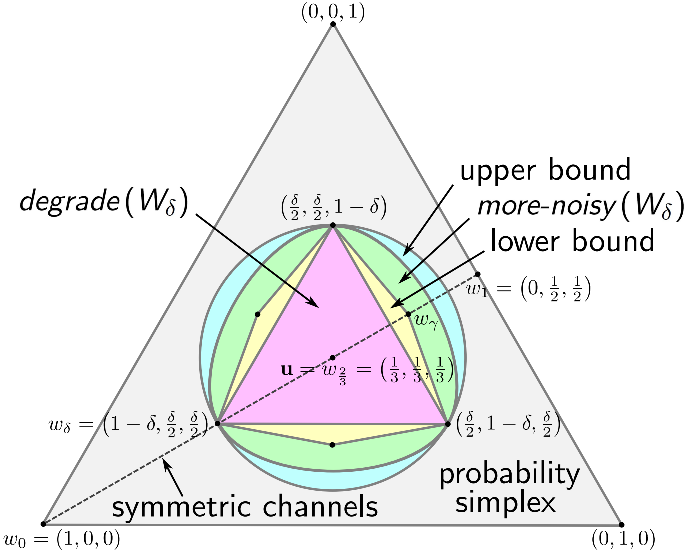

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- Log-Sobolev inequality with constant $\alpha \geq 0$:
For every $f \in \mathbb{R}^q$ such that $f^T f = q$:

$$D(f^2 \mathbf{u} \,||\, \mathbf{u}) = \frac{1}{q} \sum_{i=1}^{q} f_i^2 \log\left(f_i^2\right) \leq \frac{1}{\alpha} \mathcal{E}_V(f, f) \, .$$

Logarithmic Sobolev Inequalities

- Consider irreducible Markov chain V with uniform stationary pmf $\mathbf{u}$ on state space of size q.
- Dirichlet form $\mathcal{E}_V : \mathbb{R}^q \times \mathbb{R}^q \rightarrow [0, \infty)$

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- Log-Sobolev constant – largest α satisfying log-Sobolev inequality.

Comparison of Dirichlet Forms

- Standard Dirichlet form:

$$\mathcal{E}_{\text{std}}(f, f) \triangleq \mathbb{VAR}_{\mathbf{u}}(f) = \sum_{i=1}^{q} \frac{1}{q} f_i^2 - \left(\sum_{i=1}^{q} \frac{1}{q} f_i \right)^2$$

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- For standard Dirichlet form, $\mathcal{E}_{\mathsf{std}}(f,f) \triangleq \mathbb{VAR}_{\mathbf{u}}(f)$, log-Sobolev constant known [DSC96]:

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- $W_\delta \succeq_{\text{ln}} V \Rightarrow$ log-Sobolev inequality for V:

$$D(f^2\mathbf{u}\,||\,\mathbf{u}) \leq \frac{(q-1)\log(q-1)}{\delta\,(q-2)}\,\mathcal{E}_V(f,f)$$

for every $f \in \mathbb{R}^q$ satisfying $f^T f = q$.

Outline

Maximal Correlation and Contraction Coefficients

Definition (Maximal Correlation [Hir35, Geb41, Sar58, Rén59])

Maximal correlation between random variables $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ is:

$$\rho_{\max}(X; Y) \triangleq \max_{f, g} \mathbb{E}[f(X)g(Y)]$$

where maximization is over all $f : \mathcal{X} \to \mathbb{R}$ and $g : \mathcal{Y} \to \mathbb{R}$ such that $\mathbb{E}[f(X)] = \mathbb{E}[g(Y)] = 0$ and $\mathbb{E}\left[f(X)^2\right] = \mathbb{E}\left[g(Y)^2\right] = 1$.

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Conditional Expectation Operators

Fix bivariate distribution $P_{X,Y}$ such that $P_X > 0$ and $P_Y > 0$.

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Hilbert Spaces:

$\mathcal{L}^2(\mathcal{X}, P_X) \triangleq \left\{f : \mathcal{X} \rightarrow \mathbb{R} \,\middle|\, \mathbb{E}\left[f(X)^2\right] < +\infty\right\}$ with inner product:

$$\forall f, f' \in \mathcal{L}^2(\mathcal{X}, P_X), \ \ \langle f, f' \rangle_{P_X} \triangleq \mathbb{E}\left[f(X) f'(X)\right].$$

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$C : \mathcal{L}^2(\mathcal{X}, P_X) \to \mathcal{L}^2(\mathcal{Y}, P_Y)$ maps $f \in \mathcal{L}^2(\mathcal{X}, P_X)$ to $C(f) \in \mathcal{L}^2(\mathcal{Y}, P_Y)$:

$$(C(f))(y) \triangleq \mathbb{E}[f(X) | Y = y].$$

Singular Value Decomposition (SVD)

SVD of Conditional Expectation Operator: For $1 \leq i \leq \min\{|\mathcal{X}|, |\mathcal{Y}|\}$,

$$C(f_i) = \sigma_i g_i$$

- $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min\{|\mathcal{X}|,|\mathcal{Y}|\}} \geq 0$ are singular values,
- $\{f_1, \ldots, f_{|\mathcal{X}|}\} \subseteq \mathcal{L}^2(\mathcal{X}, P_X)$ are right singular vectors,
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- **Operator Norm:** $\|C\|_{\text{op}} = \sigma_1 = 1$, and corresponding singular vectors are $f_1 = \mathbf{1}$ and $g_1 = \mathbf{1}$.

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- **Courant-Fischer-Weyl:** For $2 \leq k \leq \min\{|\mathcal{X}|, |\mathcal{Y}|\}$,

$$\sigma_k = \mathbb{E}[f_k(X)g_k(Y)] = \max_{f,g} \mathbb{E}[f(X)g(Y)]$$

where maximization is over unit-norm $f \in \text{span}(f_1, \ldots, f_{k-1})^\perp$ and $g \in \text{span}(g_1, \ldots, g_{k-1})^\perp$.

Representation of Conditional Expectation Operators

Consider $C = \mathbb{E}_{P_{X|Y}}[\cdot|Y] : \mathcal{L}^2(\mathcal{X}, Q_X) \to \mathcal{L}^2(\mathcal{Y}, P_Y)$ with operator norm:

$$\|C\|^2_{Q_X \to P_Y} \triangleq \max_{\substack{f \in \mathcal{L}^2(\mathcal{X}, Q_X): \\ \mathbb{E}_{Q_X}[f(X)^2] = 1}} \mathbb{E}_{P_Y}\Big[\mathbb{E}_{P_{X|Y}}[f(X)|Y]^2\Big].$$

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Prop (Inner Product for Contraction Property)

- $\min_{Q_X} \|C\|_{Q_X \to P_Y}^2 = \|C\|_{P_X \to P_Y}^2 = 1$.

Remark: $Q_X^* = P_X$ is *only* inner product that makes C contractive.

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- For all Q_X, $\|C\|^2_{Q_X \to P_Y} - 1 \geq \chi^2(P_X||Q_X)$.

Remark: $Q_X^* = P_X$ is *only* inner product that makes C contractive.

Modal Decomposition

Theorem (Modal Decomposition [Hir35, Lan58])

- Modal decomposition of bivariate distribution:

$$P_{X,Y}(x,y) = P_X(x)\,P_Y(y)\left(1 + \sum_{i=2}^{\min\{|\mathcal{X}|,|\mathcal{Y}|\}} \sigma_i\, f_i(x)\, g_i(y)\right)$$

where $\{f_i\}$,$\{g_i\}$ are singular vectors of C, and $\sigma_i = \mathbb{E}[f_i(X)g_i(Y)]$ are singular values.

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- Modal decomposition of mutual χ^2-information:

$$I_{\chi^2}(X;Y) \triangleq \chi^2(P_{X,Y}||P_X P_Y) = \sum_{i=2}^{\min\{|\mathcal{X}|,|\mathcal{Y}|\}} \sigma_i^2 \,.$$

Outline

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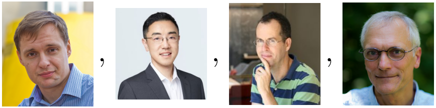
$\mathcal{X} = \{$  $, \ldots \}$

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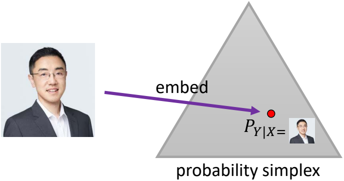
"Natural" Embedding: Represent each $x \in \mathcal{X}$ using conditional distribution $P_{Y|X=x} \in \mathbb{R}^{|\mathcal{Y}|}$.



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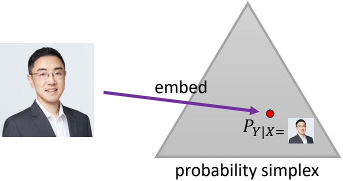

Dimensionality Reduction:
$|\mathcal{Y}|$ is large!
Reduce dimension of embedding.

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Modal Decomposition Embedding:

$$P_{Y|X=x} = P_Y + \sum_{i=2}^{\min\{|\mathcal{X}|,|\mathcal{Y}|\}} \sigma_i f_i(x) \, (g_i \cdot P_Y)$$

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Modal Decomposition Embedding: (when σ_{k+2} small)

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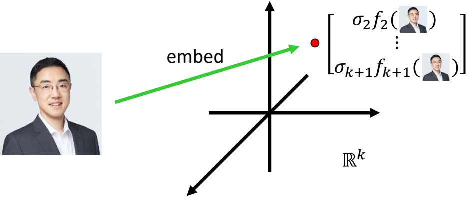

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$$= \left\| \zeta_{\min\{|\mathcal{X}|,|\mathcal{Y}|\}-1}(x) - \zeta_{\min\{|\mathcal{X}|,|\mathcal{Y}|\}-1}(x') \right\|_2^2$$

$$\approx \left\| \zeta_k(x) - \zeta_k(x') \right\|_2^2$$

Outline

Extended Alternating Conditional Expectations Algorithm

Require: joint pmf $P_{X,Y}$, number of dominant modes k

Remarks:

- Orthogonal iteration method [GvL96]

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1. **Initialization:** Randomly choose $\underline{r}_k : \mathcal{X} \to \mathbb{R}^k$.

Repeat:

2. Center and whiten $\underline{r}_k$ to obtain $\hat{r}_k$:

$$\mathbb{E}[\hat{r}_k(X)] = \mathbf{0} \text{ and } \mathbb{E}\left[\hat{r}_k(X)\hat{r}_k(X)^T\right] = I.$$

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Remarks:

- Orthogonal iteration method [GvL96]

Extended Alternating Conditional Expectations Algorithm

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- $k = 1$ case: alternating conditional expectations (ACE) algorithm for regression [BF85]

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- **Sample Version:**
  Center and update steps use operator $\hat{C}_n : \mathcal{L}^2(\mathcal{X}, P_X) \to \mathcal{L}^2(\mathcal{Y}, P_Y)$ that maps $f \in \mathcal{L}^2(\mathcal{X}, P_X)$ to $\hat{C}_n(f) \in \mathcal{L}^2(\mathcal{Y}, P_Y)$:

$$(\hat{C}_n(f))(y) \triangleq \frac{\hat{P}^n_Y(y)}{P_Y(y)} \mathbb{E}_{\hat{P}^n_{X|Y}}[f(X)|Y = y] - \mathbb{E}_{P_X}[f(X)] .$$

Sample Complexity Analysis

- Let $\hat{C}_n$ have singular values $\hat{\sigma}_2 \geq \cdots \geq \hat{\sigma}_{\max\{|\mathcal{X}|,|\mathcal{Y}|\}+1} \geq 0$ with right singular vectors $\{\hat{f}_2, \ldots, \hat{f}_{|\mathcal{X}|+1}\} \subseteq \mathcal{L}^2(\mathcal{X}, P_X)$.

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- **Convergence of "rank k approximation" of χ^2-information:**

$$\sum_{i=2}^{k+1} \mathbb{E}_{P_Y}\left[\left(\tilde{C}(\hat{f}_i)\right)(Y)^2\right] \xrightarrow{P} \sum_{i=2}^{k+1} \sigma_i^2$$

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- **Ky Fan k-Norm Estimation:** For every $0 \leq t \leq \frac{1}{\delta}\sqrt{\frac{k}{2}}$:

$$\mathbb{P}\left(\left|\|\hat{C}_n\|_{(k)} - \|\tilde{C}\|_{(k)}\right| \geq t\right) \leq \exp\left(\frac{1}{4} - \frac{n\delta^2 t^2}{8k}\right)$$

- **Singular Vector Estimation:** For every $0 \leq t \leq 4k$:

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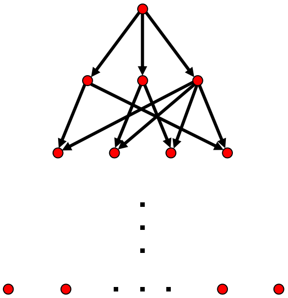
Remark: n grows with k

Outline

Broadcasting on Bounded Indegree DAGs

- Fix infinite directed acyclic graph (DAG) with single source node.



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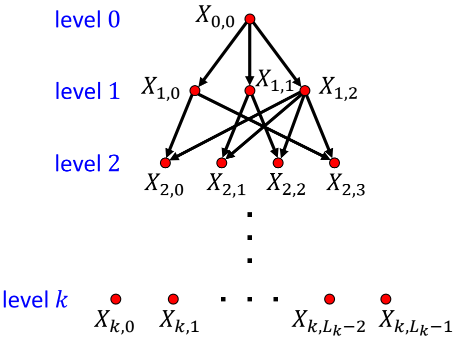

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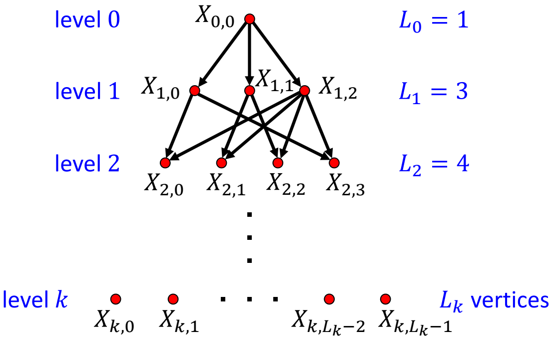

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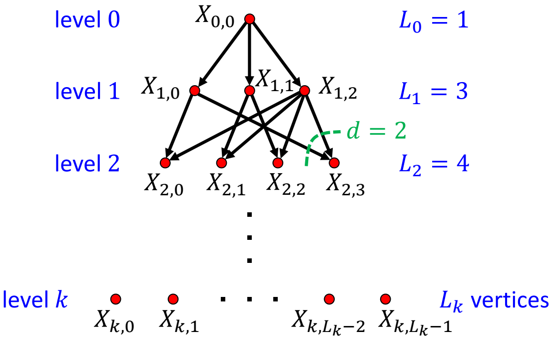

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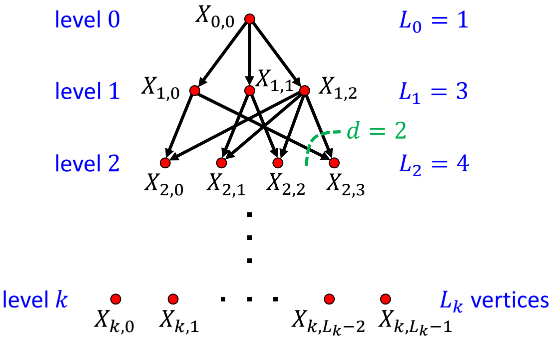

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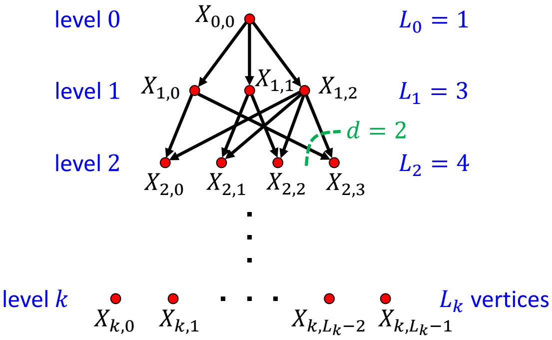

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- Nodes combine inputs with *d*-ary Boolean functions.
- This defines joint distribution of $\{X_{k,j}\}$.

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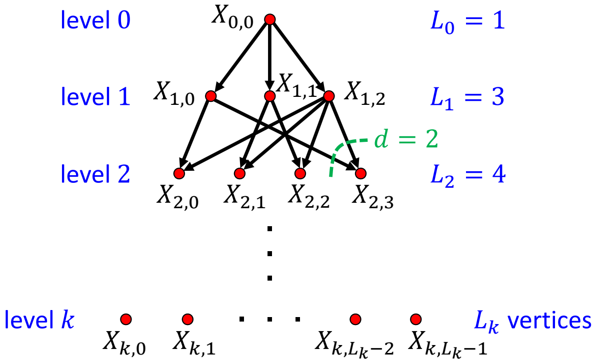
- **Can we decode X_0 from X_k as $k \to \infty$?**



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- Broadcasting $\Leftrightarrow$ TV distance contraction.

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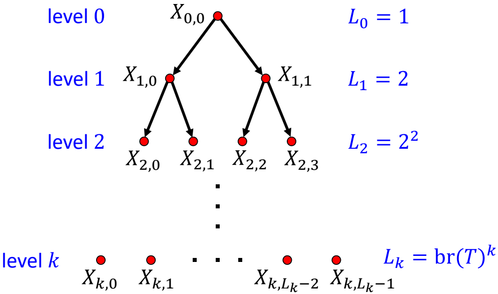
Motivation: Broadcasting on Trees

Fix tree T with $d = 1$, identity processing, and branching number $\text{br}(T)$.



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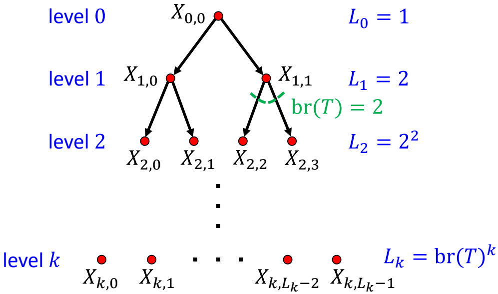

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Theorem (Phase Transition for Trees [KS66, BRZ95, EKPS00])

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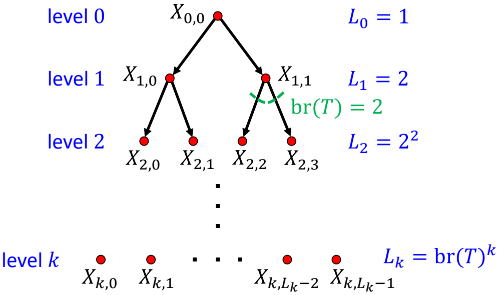

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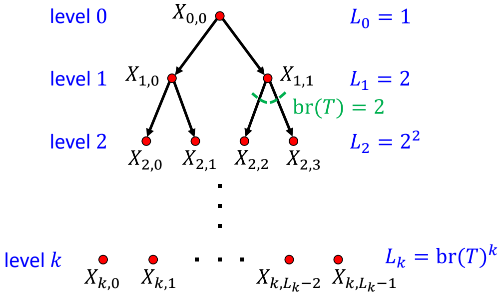

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Observations:

- L_k sub-exponential $\Rightarrow \text{br}(T) = 1$ and reconstruction impossible

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Idea: Contract $\eta_{\text{KL}}(\text{BSC}(\delta))^k = (1-2\delta)^{2k}$ along $\text{br}(T)^k$ paths [ES99].

Observations:

- L_k sub-exponential $\Rightarrow \text{br}(T) = 1$ and reconstruction impossible
- $d > 1 \Rightarrow$ information fusion at nodes

Can we broadcast with sub-exponential L_k when $d > 1$?

Motivation: Broadcasting on Trees

Fix tree T with $d = 1$, identity processing, and branching number $\text{br}(T)$.

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Can we broadcast with sub-exponential L_k when $d > 1$?

Yes, we can broadcast with $L_k = \Theta(\log(k))$!

Outline

Random DAG Model

- Fix $\{L_k\}$ and $d > 1$.

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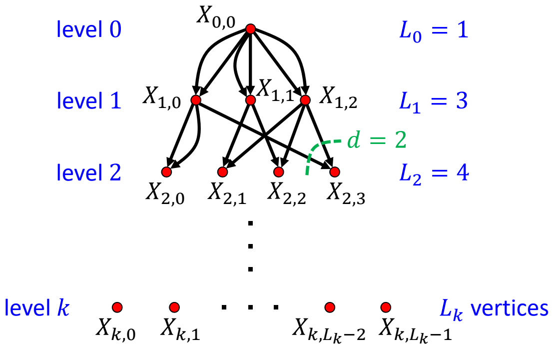
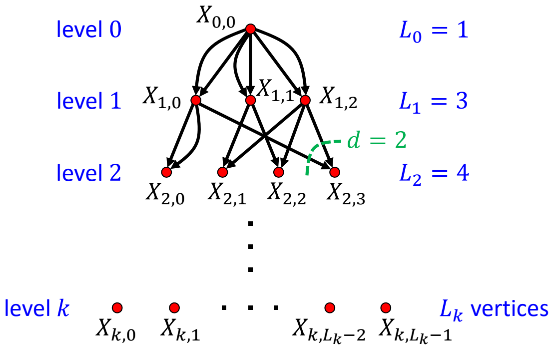


Random DAG Model

- Fix $\{L_k\}$ and $d > 1$.
- For each node $X_{k,j}$, randomly and independently select d parents from level $k-1$ (with repetition).
- This defines random DAG G.
- Let $P_{\mathsf{ML}}^{(k)}(G)$ be ML decoding probability of error for DAG G, and define $\sigma_k \triangleq \frac{1}{L_k}\sum_j X_{k,j}$ which is sufficient statistic of X_k for $\sigma_0 = X_{0,0}$.



Random DAG with Majority Processing

Theorem (Phase Transition for $d \geq 3$)

Consider random DAG model with $d \geq 3$ and majority processing (with ties broken randomly). Let $\delta_{\text{maj}} \triangleq \frac{1}{2} - \frac{2^{d-2}}{\lceil d/2 \rceil \binom{d}{\lceil d/2 \rceil}}$.

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where $\hat{S}_k \triangleq \mathbb{1}\left\{\sigma_k \geq \frac{1}{2}\right\}$ is majority decoder.

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Remarks:

- $\delta_{\text{maj}} = \frac{1}{6}$ for $d = 3$ appears in reliable computation [vNe56, HW91].
- δ_{maj} for odd $d \geq 3$ also relevant in reliable computation [ES03].
- δ_{maj} for $d \geq 3$ relevant in recursive reconstruction on trees [Mos98].

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Questions:

- Broadcasting possible with sub-logarithmic L_k?
- Broadcasting possible when $\delta > \delta_{\text{maj}}$ with other processing functions?
- What about $d = 2$?

Optimality of Logarithmic Layer Size Growth

Broadcasting possible with sub-logarithmic L_k?

Prop (Layer Size Impossibility Result)

For any deterministic DAG, if:

$$L_k \leq \frac{\log(k)}{d \log\left(\frac{1}{2\delta}\right)},$$

then reconstruction impossible for all processing functions:

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No, broadcasting impossible with sub-logarithmic L_k!

Partial Converse Results

Broadcasting possible when $\delta > \delta_{\text{maj}}$ with other processing functions?

Prop (Single Vertex Reconstruction)

Consider random DAG model with $d \geq 3$.

- If $\delta \in (0, \delta_{\text{maj}})$, $L_k \geq C(\delta, d) \log(k)$, and processing functions are majority, then single vertex reconstruction possible:

$$\limsup_{k \to \infty} \mathbb{P}(X_{k,0} \neq X_{0,0}) < \frac{1}{2} .$$

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- If $\delta \in \left[\delta_{\text{maj}}, \frac{1}{2}\right)$, d is odd, $\lim_{k\to\infty} L_k = \infty$, and $\inf_{n\geq k} L_n = O\left(d^{2k}\right)$, then single vertex reconstruction impossible for all processing functions:
$$\lim_{k\to\infty} \mathbb{E}\left[\left\|P_{X_{k,0}|G,X_{0,0}=1} - P_{X_{k,0}|G,X_{0,0}=0}\right\|_{\text{TV}}\right] = 0\,.$$

Remark: Converse uses reliable computation results [HW91, ES03].

Partial Converse Results

Broadcasting possible when $\delta > \delta_{\text{maj}}$ with other processing functions?

Prop (Information Percolation [ES99])

For any deterministic DAG, if:

$$\delta > \frac{1}{2} - \frac{1}{2\sqrt{d}} \qquad \text{and} \qquad L_k = o\left(\frac{1}{\left((1-2\delta)^2 d\right)^k}\right)$$

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Random DAG with NAND Processing

What about $d = 2$?

Theorem (Phase Transition for $d = 2$)

Consider random DAG model with $d = 2$ and NAND processing functions. Let $\delta_{\text{nand}} \triangleq \frac{3-\sqrt{7}}{4}$.

- Suppose $\delta \in (0, \delta_{\text{nand}})$. Then, there exist $C(\delta) > 0$ and $t(\delta) \in (0, 1)$ such that if $L_k \geq C(\delta)\log(k)$, then reconstruction possible:

$$\lim_{k\to\infty} \mathbb{E}\left[P_{\text{ML}}^{(k)}(G)\right] \leq \limsup_{k\to\infty} \mathbb{P}\left(\hat{T}_{2k} \neq X_{0,0}\right) < \frac{1}{2}$$

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- Suppose $\delta \in \left(\delta_{\text{nand}}, \frac{1}{2}\right)$. Then, there exist $D(\delta), E(\delta) > 1$ such that if $L_k = o\left(D(\delta)^k\right)$ and $\liminf_{k\to\infty} L_k > E(\delta)$, then reconstruction impossible:

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Remark: δ_{nand} appears in reliable computation [EP98, Ung07].

Existence of DAGs where Broadcasting is Possible

Probabilistic Method:

Random DAG broadcasting $\Rightarrow$ DAG where reconstruction possible exists.

Existence of DAGs where Broadcasting is Possible

Probabilistic Method:

Random DAG broadcasting $\Rightarrow$ DAG where reconstruction possible exists. For example:

Corollary (Existence of Deterministic Broadcasting DAGs)

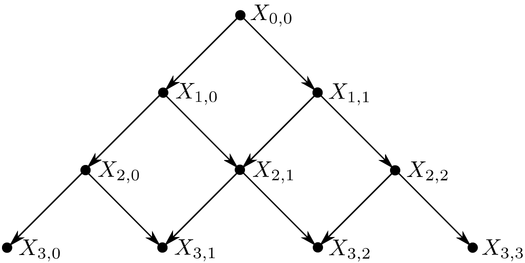
For every $d \geq 3$, $\delta \in (0, \delta_{\text{maj}})$, and $L_k \geq C(\delta, d) \log(k)$, there exists DAG with majority processing functions such that reconstruction possible:

$$\lim_{k \to \infty} P_{\text{ML}}^{(k)} < \frac{1}{2} .$$

Outline

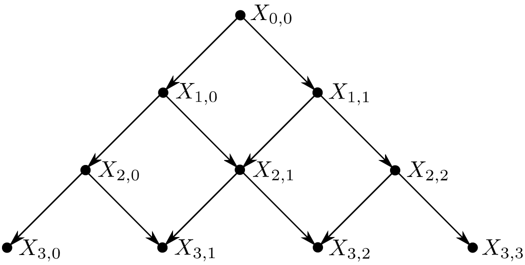
2D Regular Grid Model

- DAG is 2D regular grid with $L_k = k + 1$.



2D Regular Grid Model

- DAG is 2D regular grid with $L_k = k + 1$.
- Side nodes use identity processing.
- Other nodes use common Boolean processing function.



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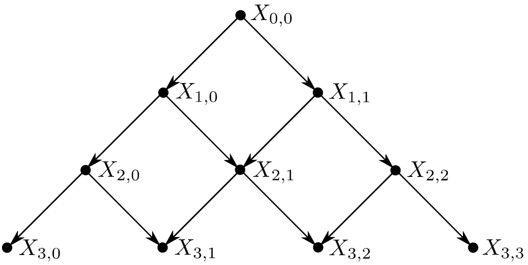

Conjecture: For all $\delta \in \left(0, \frac{1}{2}\right)$ and common processing functions, reconstruction impossible on 2D regular grid model.

Motivation: "Positive rates conjecture" on ergodicity of simple 1D probabilistic cellular automata.

Impossibility of Broadcasting

Theorem (2D Regular AND Grid)

For all $\delta \in \left(0, \frac{1}{2}\right)$, reconstruction impossible on 2D regular grid model with AND processing:

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Theorem (2D Regular XOR Grid)

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Main Contributions:

- Properties of contraction coefficients

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- Broadcasting impossible in 2D regular grids with AND/XOR processing

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Thank You!