Less Noisy Domination by Symmetric Channels

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Outline

Preliminaries

- probability distributions – *row* vectors

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- channels (conditional distributions) – *row stochastic* matrices

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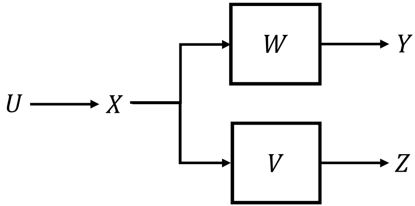
Definition (Less Noisy Preorder [Körner-Marton 1977])

$P_{Y|X} = W$ is less noisy than $P_{Z|X} = V$, denoted $W \succeq_{\text{ln}} V$, if and only if

$$I(U;Y) \geq I(U;Z)$$

for every joint distribution $P_{U,X}$ such that $U \to X \to (Y,Z)$.



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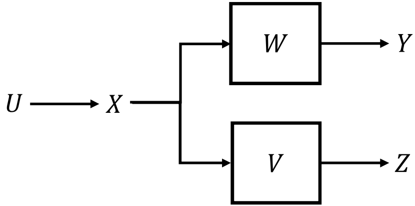
for every pair of input distributions P_X and Q_X.



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2. Sufficient conditions for $\succeq_{\ln}$ domination by symmetric channels?
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   - stronger criterion for additive noise channels

3. Why $\succeq_{\ln}$ domination by symmetric channels?
   - just because we ♥ IT
   - $\succeq_{\ln}$ domination $\Rightarrow$ log-Sobolev inequality
   - secrecy capacity

Motivation: Strong Data Processing Inequality

Data Processing Inequality:

For any channel V,

$$\forall P_X, Q_X, \;\; D(P_X || Q_X) \geq D(P_X V || Q_X V)$$

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where $\eta_{\mathsf{KL}}(V)$ – contraction coefficient:

$$\eta_{\mathsf{KL}}(V) \triangleq \sup_{P_X, Q_X} \frac{D(P_X V || Q_X V)}{D(P_X || Q_X)} \in [0, 1] \,.$$

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- **Definition:** q-ary erasure channel q-$EC(1-\eta)$
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SDPI $\Leftrightarrow$ $\succeq_{\mathsf{ln}}$ domination by erasure channel

Main Question

Given channel V, find q-ary symmetric channel W_δ with largest $\delta \in \left[0, \frac{q-1}{q}\right]$ such that $W_\delta \succeq_{\text{ln}} V$?

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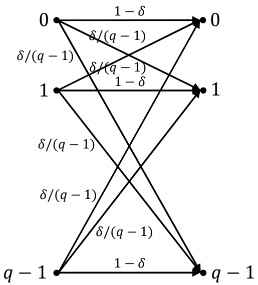
Definition (q-ary Symmetric Channel)

Channel matrix:

$$W_\delta \triangleq \begin{bmatrix} 1-\delta & \frac{\delta}{q-1} & \cdots & \frac{\delta}{q-1} \\ \frac{\delta}{q-1} & 1-\delta & \cdots & \frac{\delta}{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta}{q-1} & \frac{\delta}{q-1} & \cdots & 1-\delta \end{bmatrix}$$

where $\delta \in [0,1]$ – total crossover probability.



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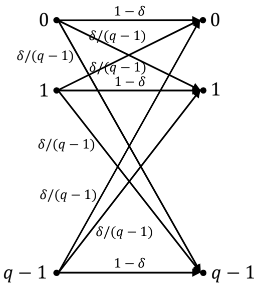

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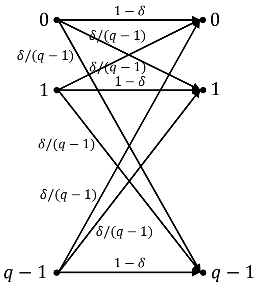

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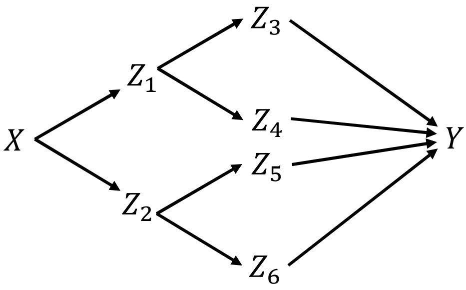
- $\forall \epsilon, \delta \in (0,1)$, $W_\delta \not\succeq_{\text{ln}} q\text{-}EC(\epsilon)$.

Less Noisy Channels in Networks

Consider general Bayesian network:



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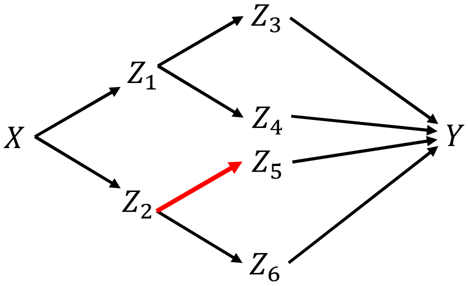

Conjecture:
Replace $P_{Z_5|Z_2}$ with less noisy channel $\Rightarrow$ $P_{Y|X}$ becomes less noisy.

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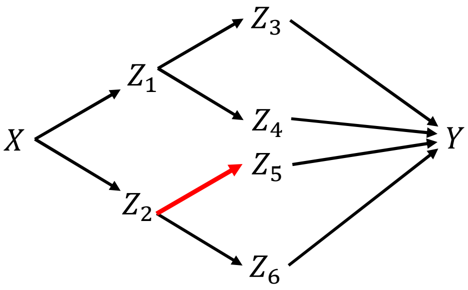

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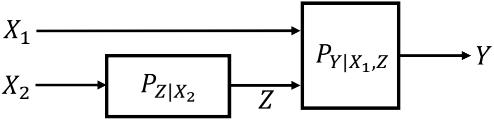
Motivation: Results of [Polyanskiy-Wu 2016] on SDPIs in networks.

Less Noisy Channels in Networks

Consider Bayesian network with binary r.v.s



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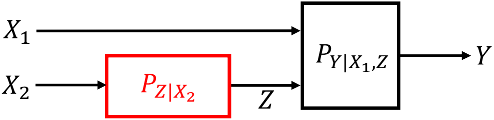

where we replace $P_{Z|X_2}$ with less noisy channel.

Can this decrease $I(X_1, X_2; Y)$**?**

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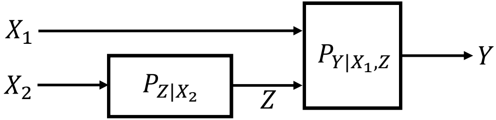
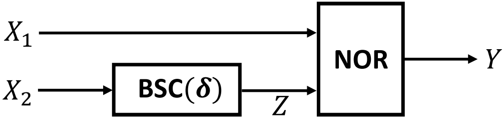


where we replace $P_{Z|X_2}$ with less noisy channel.

Can this decrease $I(X_1, X_2; Y)$? YES

Example: Let $X_1 \sim \text{Ber}\left(\frac{1}{2}\right)$ and $X_2 = 1$ *a.s.*, and let $I(\delta) = I(X_1, X_2; Y)$.



For $\delta > 0$, $\text{BSC}(0) \succeq_{\text{ln}} \text{BSC}(\delta)$, but $h(\delta/2) - h(\delta)/2 = I(\delta) > I(0) = 0$.

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χ^2-Divergence Characterization of Less Noisy

Theorem 1 (χ^2-Divergence Characterization of $\succeq_{\text{ln}}$)

Given channels W and V, $W \succeq_{\text{ln}} V$ if and only if

$$\forall P_X, Q_X, \;\; \chi^2(P_X W || Q_X W) \geq \chi^2(P_X V || Q_X V) \,.$$

Recall χ^2-divergence between P_X and Q_X:

$$\chi^2(P_X || Q_X) \triangleq \sum_{x \in \mathcal{X}} \frac{(P_X(x) - Q_X(x))^2}{Q_X(x)} \,.$$

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Proof: ($\Rightarrow$) Fix any P_X, Q_X. Recall local approximation:

$$\lim_{\lambda \to 0^+} \frac{2}{\lambda^2} D(\lambda P_X + (1-\lambda) Q_X || Q_X) = \chi^2(P_X || Q_X) \, .$$

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after taking limits.

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where $Q_X^t = \frac{t}{1+t} P_X + \frac{1}{t+1} Q_X$ for $t \in [0, \infty)$ [Choi-Ruskai-Seneta 1994].

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$$D(P_X W || Q_X W) \geq D(P_X V || Q_X V)$$

Löwner and Spectral Characterizations of Less Noisy

Theorem 1 (Equivalent Characterizations of $\succeq_{\text{ln}}$)

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$$W \succeq_{\text{ln}} V \;\Leftrightarrow\; \forall P_X, Q_X,\ \chi^2(P_X W || Q_X W) \geq \chi^2(P_X V || Q_X V)$$
$$\Leftrightarrow\; \forall P_X,\ W \text{diag}(P_X W)^{-1} W^T \succeq_{\text{PSD}} V \text{diag}(P_X V)^{-1} V^T$$
$$\Leftrightarrow\; \forall P_X,\ \rho\Big(\big(W \text{diag}(P_X W)^{-1} W^T\big)^{\dagger} V \text{diag}(P_X V)^{-1} V^T\Big) = 1$$

where $\succeq_{\text{PSD}}$ – Löwner (PSD) partial order,
$A^{\dagger}$ – Moore-Penrose pseudoinverse of A,
and $\rho(\cdot)$ – spectral radius.

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Condition for Degradation by Symmetric Channels

Given channel V, find q-ary symmetric channel W_δ with largest $\delta \in \left[0, \frac{q-1}{q}\right]$ such that $W_\delta \succeq_{\text{ln}} V$?

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Theorem 2 (Degradation by Symmetric Channels)

For channel V with common input and output alphabet, and minimum probability $\nu = \min\left\{[V]_{i,j} : 1 \leq i, j \leq q\right\}$,

$$0 \leq \delta \leq \frac{\nu}{1 - (q-1)\nu + \frac{\nu}{q-1}} \Rightarrow W_\delta \succeq_{\text{deg}} V \, .$$

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$$0 \leq \delta \leq \frac{\nu}{1-(q-1)\nu + \frac{\nu}{q-1}} \ \Rightarrow \ W_\delta \succeq_{\text{deg}} V \,.$$

Remark: Condition is tight when no further information about V known. For example, suppose

$$V = \begin{bmatrix} \nu & 1-(q-1)\nu & \nu & \cdots & \nu \\ 1-(q-1)\nu & \nu & \nu & \cdots & \nu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-(q-1)\nu & \nu & \nu & \cdots & \nu \end{bmatrix}.$$

Then, $0 \leq \delta \leq \nu/\left(1-(q-1)\nu + \frac{\nu}{q-1}\right) \ \Leftrightarrow \ W_\delta \succeq_{\text{deg}} V$.

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- q-ary symmetric channel: $P_Z = \left(1 - \delta, \frac{\delta}{q-1}, \ldots, \frac{\delta}{q-1}\right)$ for $\delta \in [0, 1]$

$$(\cdot \star P_Z) = W_\delta$$

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Domination Structure of Additive Noise Channels

Theorem 3 (More Noisy and Degradation Regions)

For W_δ with $\delta \in \left[0, \frac{q-1}{q}\right]$ and $q \geq 2$,

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Furthermore, *more-noisy* (W_δ) is closed, convex, and invariant under permutations of $(\mathcal{X}, \oplus)$.

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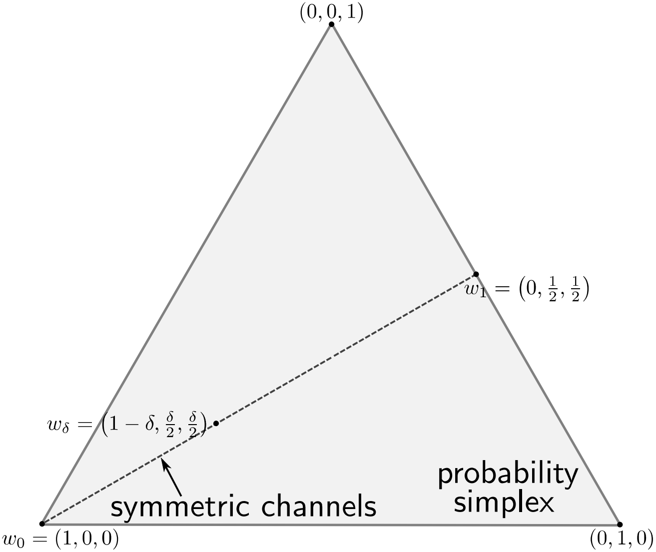

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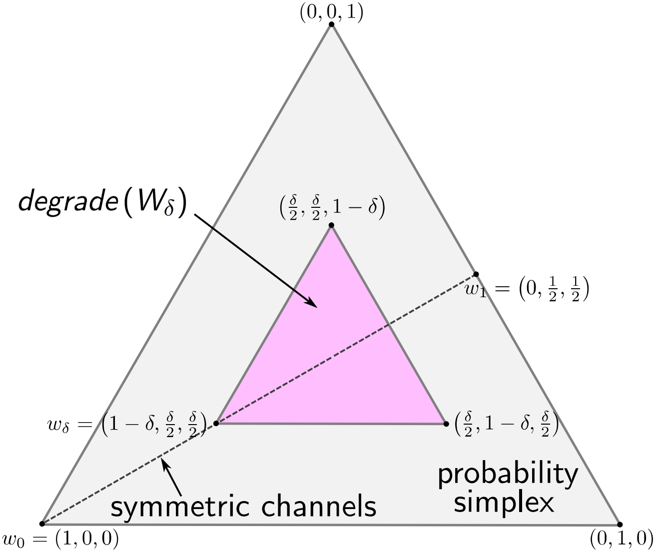

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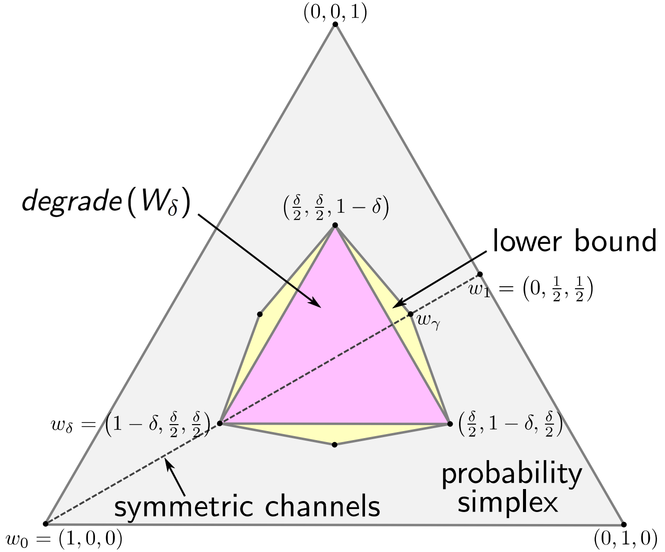

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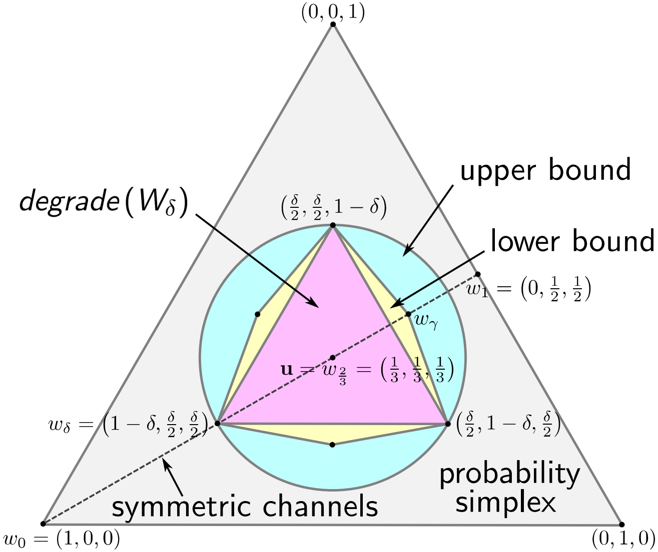

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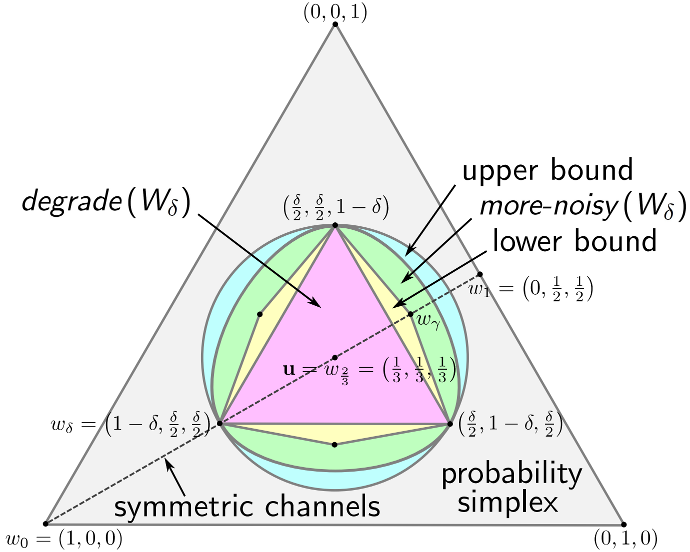

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where $\|\cdot\|_{\text{op}}$ – operator norm, and A is symmetric PSD:

$$A \triangleq \operatorname{diag}(w_\gamma)^{-\frac{1}{2}} W_\gamma W_\delta^{-1} \operatorname{diag}(w_\delta) W_\delta^{-1} W_\gamma \operatorname{diag}(w_\gamma)^{-\frac{1}{2}}$$

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- A has left eigenvector $\sqrt{w_\gamma} > 0$ with eigenvalue 1:

$$\sqrt{w_\gamma} A = \sqrt{w_\gamma}\,.$$

Proof Sketch: $co\,($rows of W_δ and $W_\gamma) \subseteq$ $\textit{more-noisy}\,(W_\delta)$

- $A \geq 0$ (entry-wise) $\Rightarrow$ largest eigenvalue of A is 1 by Perron-Frobenius theorem, because $\sqrt{w_\gamma} > 0$.

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- Verify that:

$$\min_{i,j} [A]_{i,j} \geq 0 \quad \Leftrightarrow \quad \delta \leq \gamma \leq \frac{1-\delta}{1-\delta+\frac{\delta}{(q-1)^2}} .$$

■

Outline

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- Log-Sobolev constant – largest α satisfying log-Sobolev inequality.

Comparison of Dirichlet Forms

- Standard Dirichlet form:

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Theorem 4 (Domination of Dirichlet Forms)

For channels W_δ and V with $\delta \in \left[0, \frac{q-1}{q}\right]$ and stationary pmf $\mathbf{u}$,

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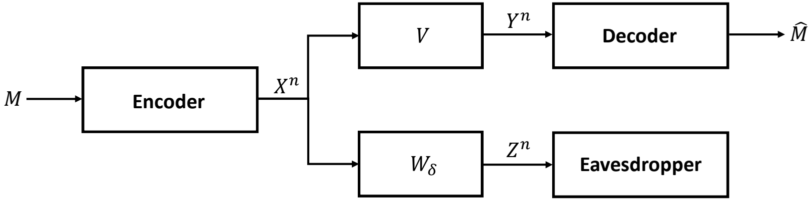
Interpretation via Wyner's Wiretap Channel



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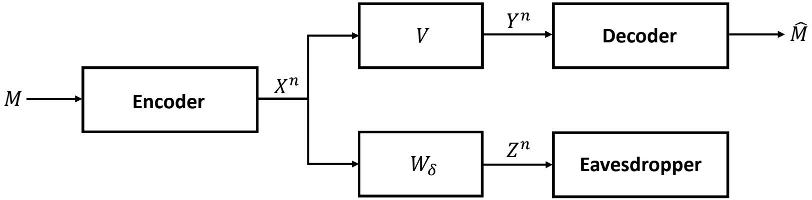

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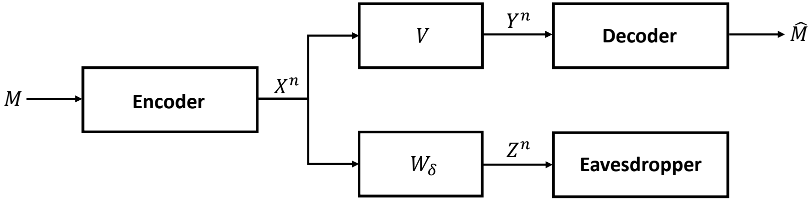

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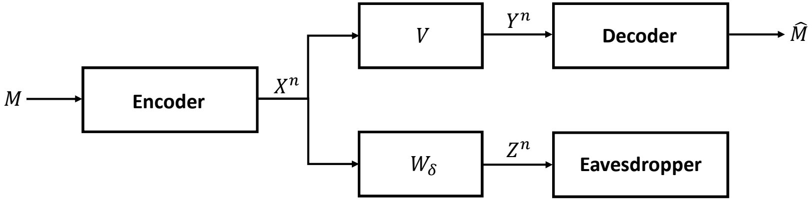

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- Finding maximally noisy $W_\delta \succeq_{\text{ln}} V$ establishes minimal noise on $P_{Z|X}$ so that secret communication feasible.

Thank You!