Broadcasting on Random Networks

Anuran Makur, Elchanan Mossel, and Yury Polyanskiy

EECS and Mathematics Departments
Massachusetts Institute of Technology

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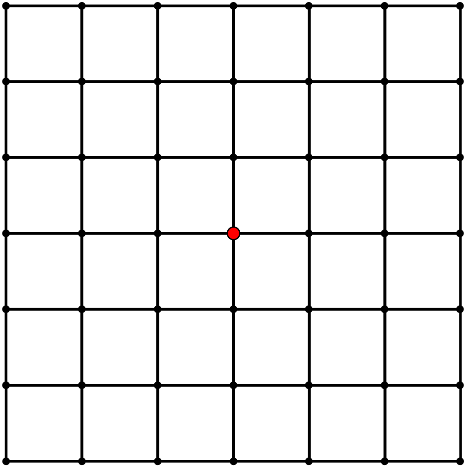
Outline

Motivation: Information Propagation in 2D Grid

- How does information spread in time?



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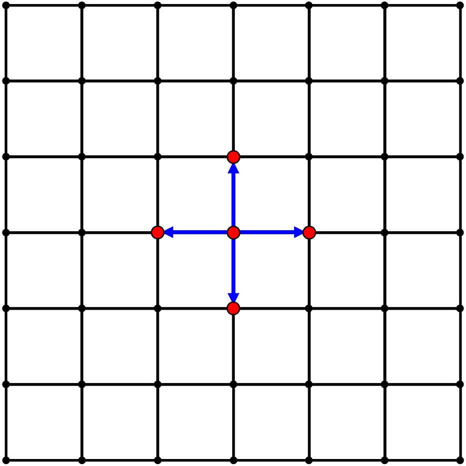

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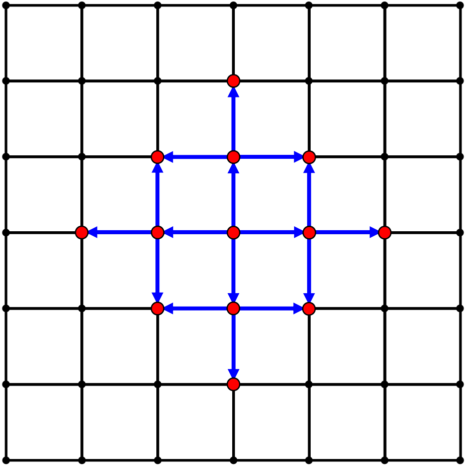

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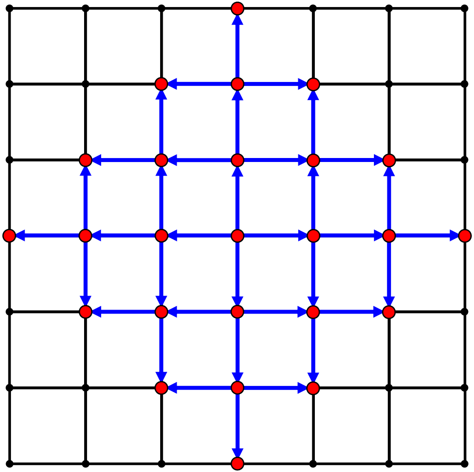

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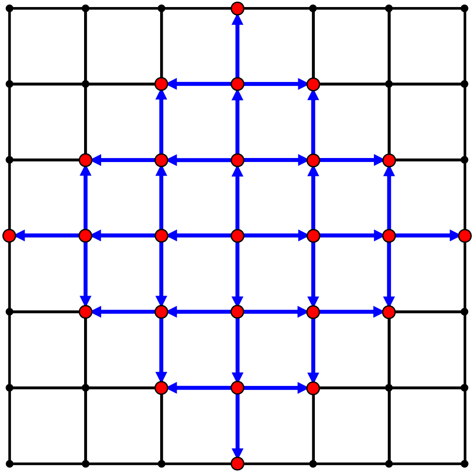
- How does information spread in time?
- Can we invent relay functions so that far boundary contains non-trivial information about the original bit?



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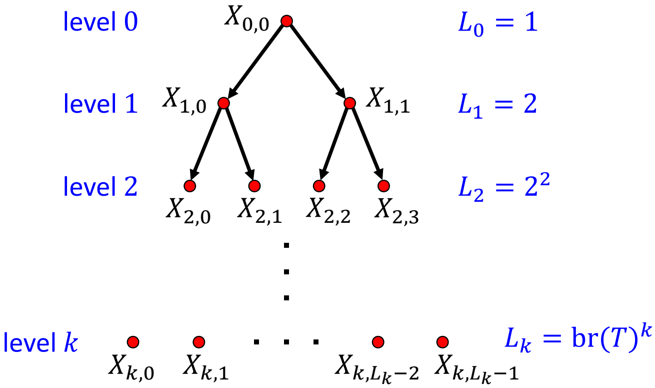
- Fix infinite tree T with branching number $\text{br}(T)$.



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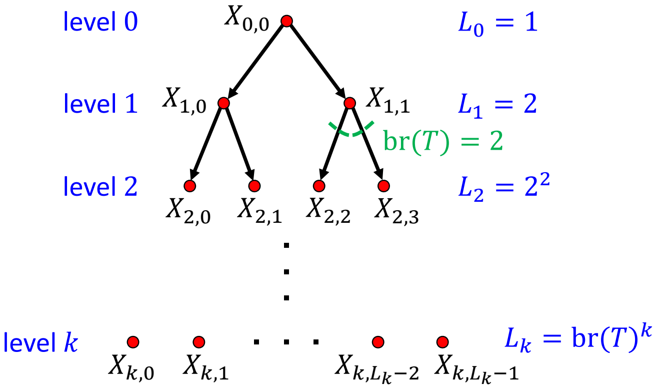

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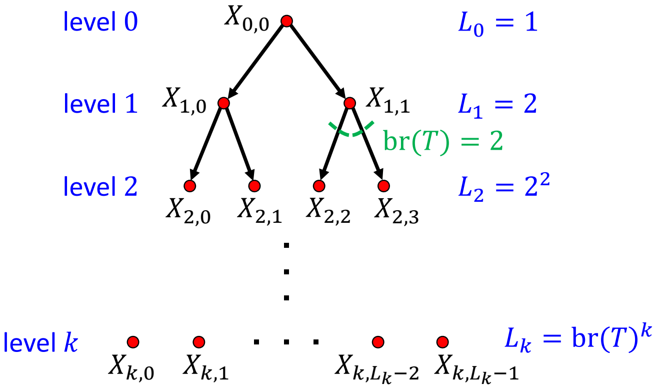

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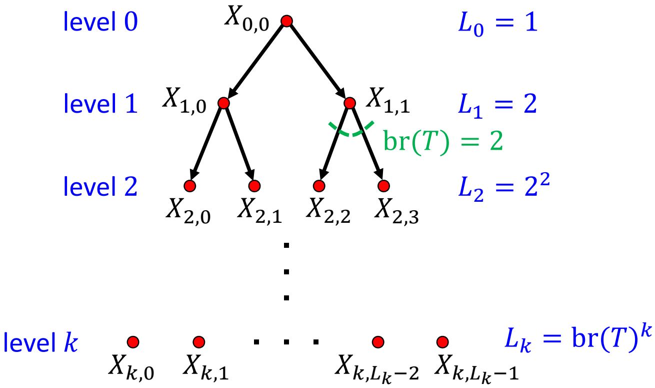

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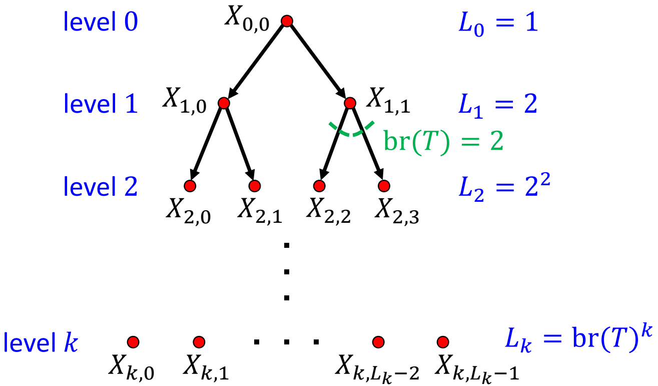

Motivation: Broadcasting on Trees

Theorem (Phase Transition for Trees [KS66, BRZ95, EKPS00])

- If $\delta < \frac{1}{2} - \frac{1}{2\sqrt{\mathrm{br}(T)}}$, then reconstruction possible: $\lim_{k\to\infty} P_{\mathrm{ML}}^{(k)} < \frac{1}{2}$.
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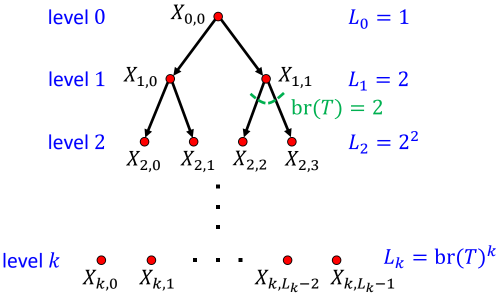

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Theorem (Phase Transition for Trees [KS66, BRZ95, EKPS00])

- If $(1-2\delta)^2 \,\mathrm{br}(T) > 1$, then reconstruction possible: $\lim_{k\to\infty} P_{\mathsf{ML}}^{(k)} < \frac{1}{2}$.
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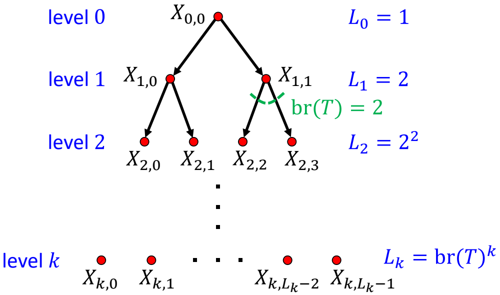
Proof Idea: Strong data processing inequality [AG76, ES99]



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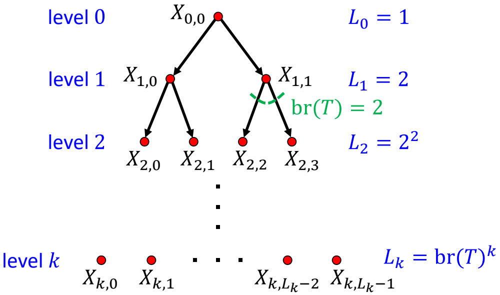

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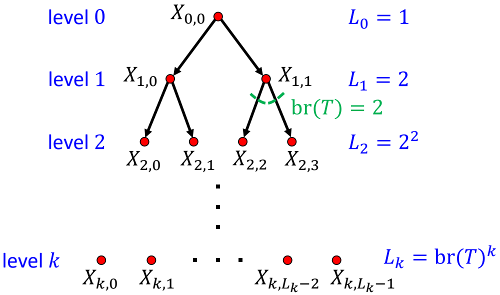

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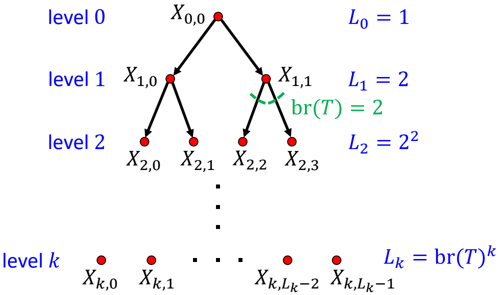
- For any $0 \le j < \text{br}(T)^k$, $I(X_{0,0};X_{k,j}) \le (1-2\delta)^{2k}$.
- $\text{br}(T)^k$ paths from X_0 to X_k: $I(X_0;X_k) \le \left(\text{br}(T)(1-2\delta)^2\right)^k$.



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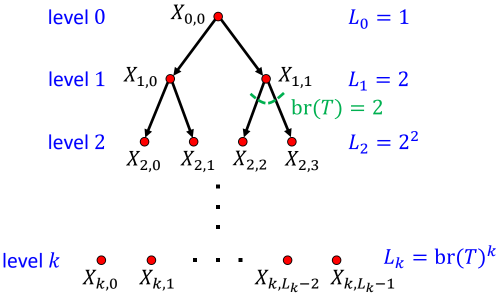
Layers grow by $\mathrm{br}(T)$ **and information contracts by** $(1-2\delta)^2$**. So, whichever effect wins determines reconstruction.**



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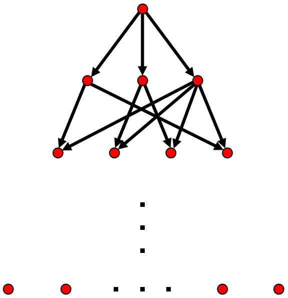
Can there be any graph with sub-exponentially growing layer sizes such that reconstruction possible?

Surprise: Yes, and in fact, even logarithmic growth suffices (doubly-exponential reduction compared to trees (!)).

But need nice loops to aggregate information.

Formal Model: Broadcasting on Bounded Indegree DAGs

- Fix infinite directed acyclic graph (DAG) with single source node.



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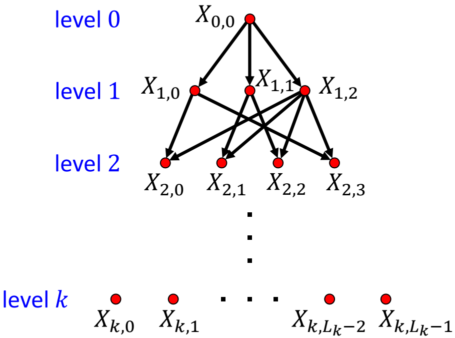

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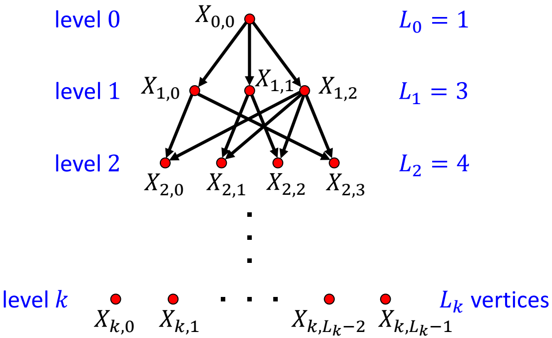

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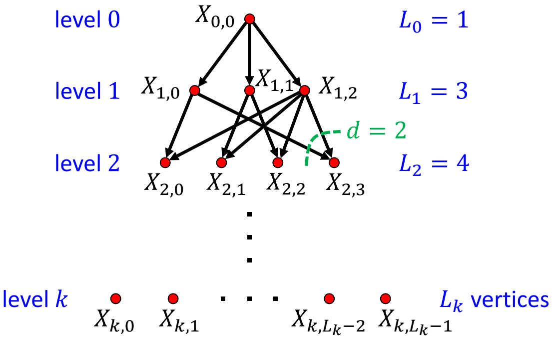

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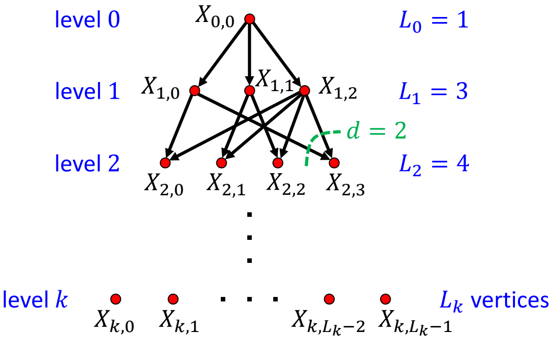

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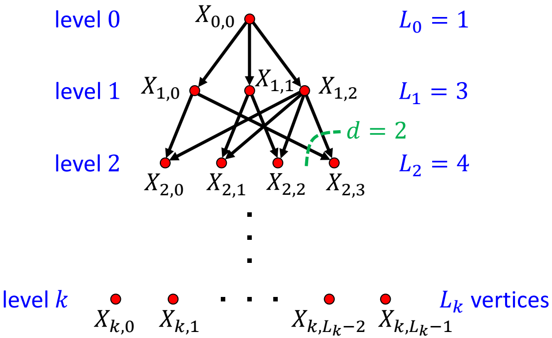

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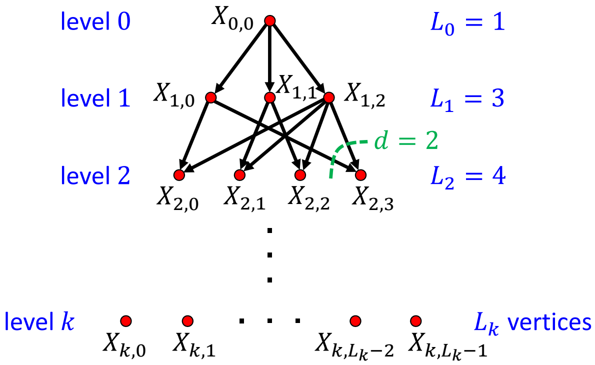
- Nodes combine inputs with *d*-ary Boolean functions.
- This defines joint distribution of $\{X_{k,j}\}$.

Broadcasting Problem

- Let $X_k \triangleq (X_{k,0}, \ldots, X_{k,L_k-1})$.
- **Can we decode X_0 from X_k as $k \to \infty$?**



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- Broadcasting/Reconstruction possible if:

$$\lim_{k\to\infty} P_{\text{ML}}^{(k)} < \frac{1}{2} \quad \Leftrightarrow \quad \lim_{k\to\infty} \left\|P_{X_k|X_0=1} - P_{X_k|X_0=0}\right\|_{\text{TV}} > 0$$

and Broadcasting/Reconstruction impossible if:

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For which δ, d, $\{L_k\}$, and Boolean processing functions is reconstruction possible?

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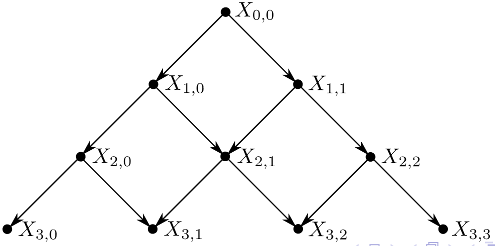
  Impossibility of broadcasting on 2D regular grid parallels ergodicity of 1D probabilistic cellular automata.



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- **Ferromagnetic Ising Models:** [BRZ95, EKPS00]
  Reconstruction impossible on *tree* $\Leftrightarrow$ Free boundary Gibbs state of Ising model on tree is extremal.

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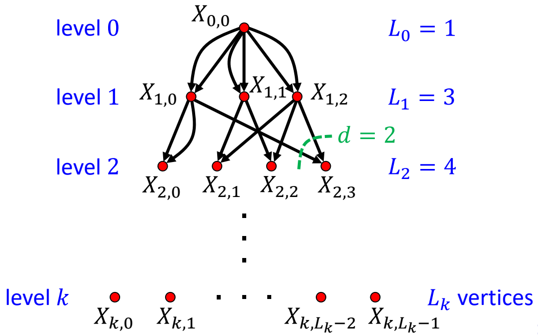

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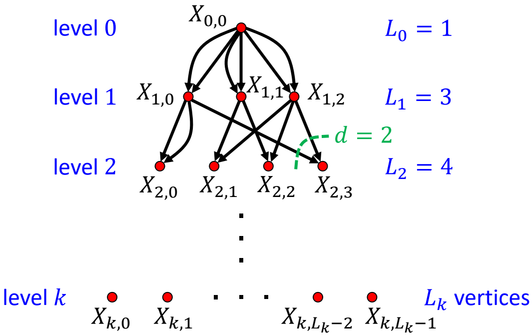

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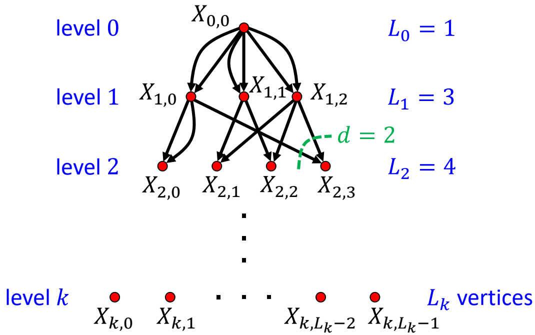
- $\sigma_k \triangleq \frac{1}{L_k}\sum_{j=0}^{L_k-1} X_{k,j}$ – sufficient statistic of X_k for $\sigma_0 = X_{0,0}$ in the absence of knowledge of G



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Theorem (Phase Transition for $d \geq 3$)

Consider random DAG model with $d \geq 3$ and majority processing (with ties broken randomly). Let $\delta_{\mathsf{maj}} \triangleq \frac{1}{2} - \frac{2^{d-2}}{\lceil d/2 \rceil \binom{d}{\lceil d/2 \rceil}}$.

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- Suppose $\delta \in (0, \delta_{\text{maj}})$. Then, there exists $C(\delta, d) > 0$ such that if $L_k \geq C(\delta, d) \log(k)$, then reconstruction possible:

$$\limsup_{k \to \infty} \mathbb{P}\left(\hat{S}_k \neq X_{0,0}\right) < \frac{1}{2}$$

where $\hat{S}_k \triangleq \mathbb{1}\left\{\sigma_k \geq \frac{1}{2}\right\}$ is majority decoder.

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$$L_k \sigma_k = \sum_{j=0}^{L_k - 1} X_{k,j} \sim \text{binomial}(L_k, g_\delta(\sigma)) .$$

- Define the cubic polynomial:

$$g_\delta(\sigma) \triangleq \mathbb{E}[\sigma_k | \sigma_{k-1} = \sigma] = \mathbb{P}(X_{k,j} = 1 | \sigma_{k-1} = \sigma)$$
$$= (\sigma * \delta)^3 + 3(\sigma * \delta)^2 (1 - \sigma * \delta) .$$

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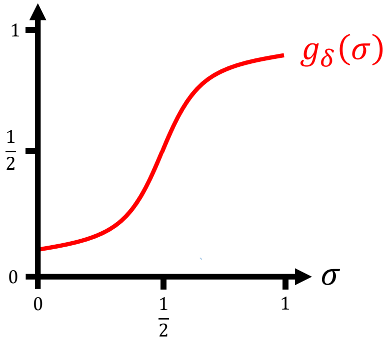

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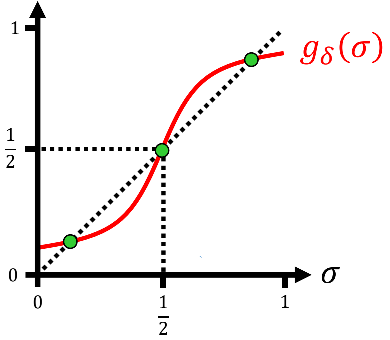
- **Fixed Point Analysis:** σ_k "concentrates" at fixed point near $X_{0,0}$

Case $\delta < \delta_{\mathsf{maj}}$: 3 fixed points



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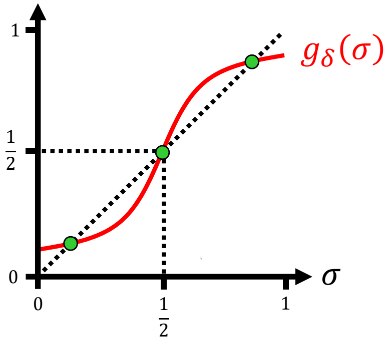

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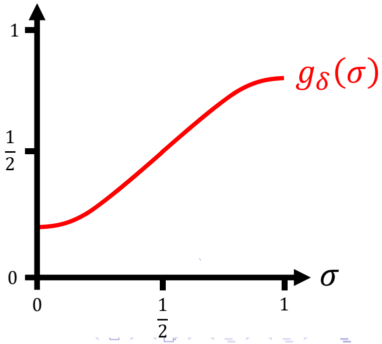

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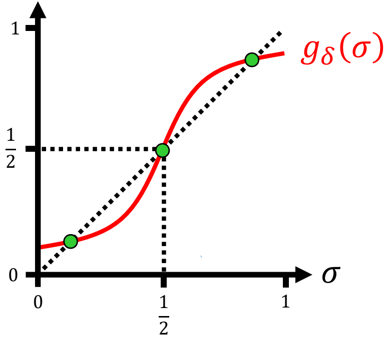

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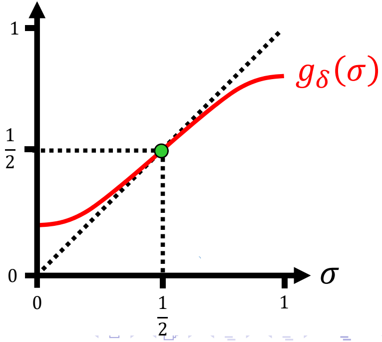

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- **Concentration:** For large k, $\sigma_k \approx g_\delta(\sigma_{k-1})$ given σ_{k-1}.
- **Converse** uses *key property*: $\text{Lip}(g_\delta) \leq 1 \Leftrightarrow g_\delta$ has unique fixed point.

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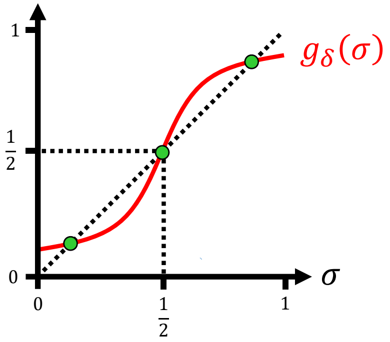

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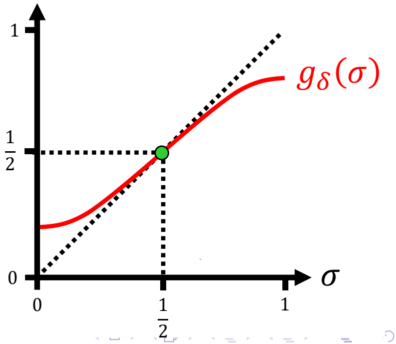

Random DAG with Majority Processing

Theorem (Phase Transition for $d \geq 3$)

Consider random DAG model with $d \geq 3$ and majority processing (with ties broken randomly). Let $\delta_{\mathsf{maj}} \triangleq \frac{1}{2} - \frac{2^{d-2}}{\lceil d/2 \rceil \binom{d}{\lceil d/2 \rceil}}$.

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Remarks:

- $\delta_{\text{maj}} = \frac{1}{6}$ for $d = 3$ appears in reliable computation [vNe56, HW91].
- δ_{maj} for odd $d \geq 3$ also relevant in reliable computation [ES03].
- δ_{maj} for $d \geq 3$ relevant in recursive reconstruction on trees [Mos98].

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Questions:

- Broadcasting possible with sub-logarithmic L_k?
- Broadcasting possible when $\delta > \delta_{\text{maj}}$ with other processing functions?
- What about $d = 2$?

Optimality of Logarithmic Layer Size Growth

Broadcasting possible with sub-logarithmic L_k?

Proposition (Layer Size Impossibility Result)

For any deterministic DAG, if:

$$L_k \leq \frac{\log(k)}{d \log\left(\frac{1}{2\delta}\right)} ,$$

then reconstruction impossible for all processing functions:

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No, broadcasting impossible with sub-logarithmic L_k!

Partial Converse Results

Broadcasting possible when $\delta > \delta_{\text{maj}}$ with other processing functions?

Proposition (Single Vertex Reconstruction)

Consider random DAG model with $d \geq 3$.

- If $\delta \in (0, \delta_{\text{maj}})$, $L_k \geq C(\delta, d) \log(k)$, and processing functions are majority, then single vertex reconstruction possible:

$$\limsup_{k \to \infty} \mathbb{P}(X_{k,0} \neq X_{0,0}) < \frac{1}{2} .$$

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- If $\delta \in \left[\delta_{\text{maj}}, \frac{1}{2}\right)$, d is odd, $\lim_{k \to \infty} L_k = \infty$, and $\inf_{n \geq k} L_n = O\left(d^{2k}\right)$, then single vertex reconstruction impossible for all processing functions (which may be graph dependent):

$$\lim_{k \to \infty} \mathbb{E}\left[\left\| P_{X_{k,0}|G, X_{0,0}=1} - P_{X_{k,0}|G, X_{0,0}=0} \right\|_{\text{TV}}\right] = 0\,.$$

Remark: Converse uses reliable computation results [HW91, ES03].

Partial Converse Results

Broadcasting possible when $\delta > \delta_{\text{maj}}$
with other processing functions?

Proposition (Information Percolation [ES99, PW17])

For any deterministic DAG, if:

$$\delta > \frac{1}{2} - \frac{1}{2\sqrt{d}} \qquad \text{and} \qquad L_k = o\left(\frac{1}{\left((1-2\delta)^2 d\right)^k}\right)$$

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Random DAG with NAND Processing

What about $d = 2$?

Theorem (Phase Transition for $d = 2$)

Consider random DAG model with $d = 2$ and NAND processing functions. Let $\delta_{\text{nand}} \triangleq \frac{3-\sqrt{7}}{4}$.

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Remark: δ_{nand} appears in reliable computation [EP98, Ung07].

Outline

Probabilistic Method:

Random DAG broadcasting $\Rightarrow$ DAG where reconstruction possible exists.

Existence of DAGs where Broadcasting is Possible

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For example:

Corollary (Existence of Deterministic Broadcasting DAGs)

For every $d \geq 3$, $\delta \in (0, \delta_{\mathsf{maj}})$, and $L_k \geq C(\delta, d)\log(k)$, there exists DAG with majority processing functions such that reconstruction possible:

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Can we construct such DAGs for any $\delta \in \left(0, \frac{1}{2}\right)$?

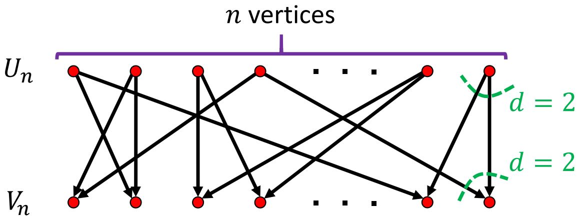
Regular Bipartite Expander Graphs

Proposition (Existence of Expander Graphs [Pin73, SS96])

For all (large) d and all sufficiently large n, there exists d-regular bipartite graph $B_n = (U_n, V_n, E_n)$ with disjoint vertex sets U_n, V_n of cardinality $|U_n| = |V_n| = n$, edge multiset E_n, and the lossless expansion property:

$$\forall S \subseteq U_n, \quad |S| = \frac{n}{d^{6/5}} \;\; \Rightarrow \;\; |\Gamma(S)| \geq \left(1 - \frac{2}{d^{1/5}}\right) d|S|$$

where $\Gamma(S) \triangleq \{v \in V_n : \exists u \in S, \, (u,v) \in E_n\}$ is neighborhood of S.



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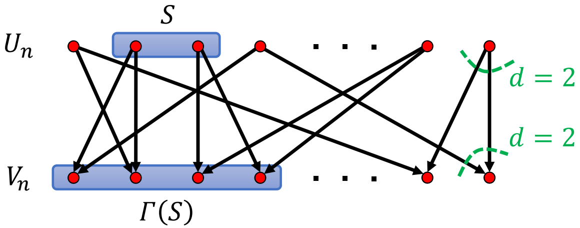

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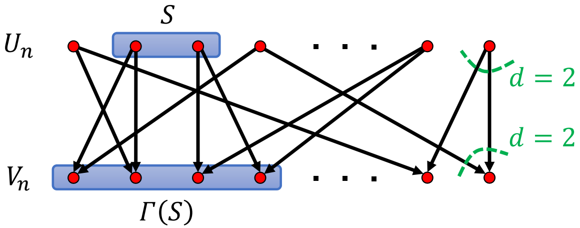
Intuition: Expander graphs are sparse, but have high connectivity.



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$$\forall \;\; r \geq 1, \;\; M^{2^{r-1}} < k \leq M^{2^r}, \quad L_k = 2^r N$$

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- Construct bounded degree deterministic "expander DAG":
  - Each $X_{1,j}$ has one edge from $X_{0,0}$.

Construction of DAGs where Broadcasting is Possible

- Fix any $\delta \in \left(0, \frac{1}{2}\right)$ and any sufficiently large odd $d = d(\delta)$.
- Fix $L_0 = 1$, $L_k = N$ for $k \in \{1, \dots, \lfloor M \rfloor\}$ where $N = N(\delta)$ sufficiently large and $M = \exp\big(N/(4d^{12/5})\big)$, and

$$\forall \ r \geq 1, \ \ M^{2^{r-1}} < k \leq M^{2^r}, \quad L_k = 2^r N$$

  such that $L_k = \Theta(\log(k))$.
- Construct bounded degree deterministic "expander DAG":
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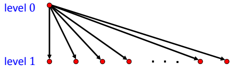
Construction of DAGs where Broadcasting is Possible

Illustration of "Expander DAG":



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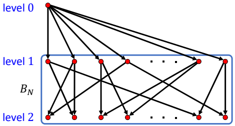

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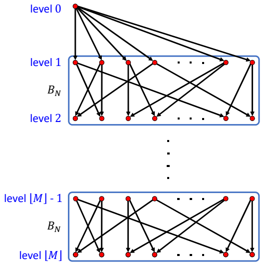

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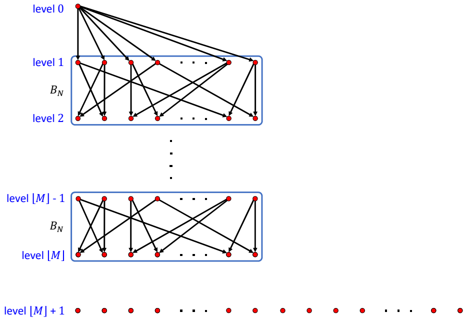

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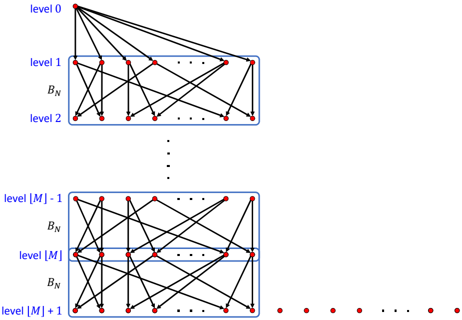

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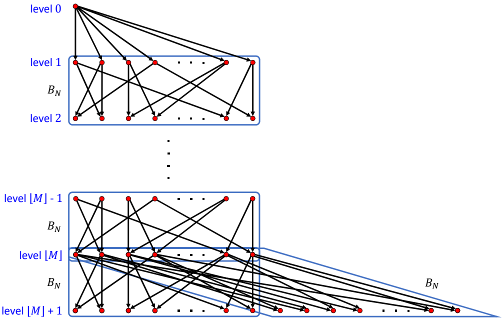

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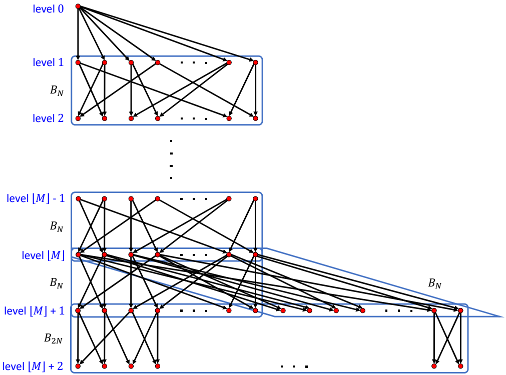

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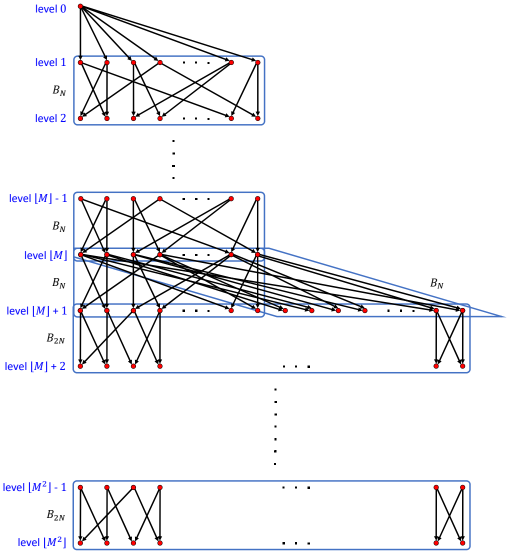

Construction of DAGs where Broadcasting is Possible

Theorem (Broadcasting in Expander DAG)

For "expander DAG" with majority processing, reconstruction possible:

$$\limsup_{k \to \infty} \mathbb{P}\left(\hat{S}_k \neq X_{0,0}\right) < \frac{1}{2}$$

where $\hat{S}_k = \mathbb{1}\left\{\sigma_k \geq \frac{1}{2}\right\}$ is majority decoder.

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- If $X_{0,0} = 0$, then $|\mathcal{S}_k|$ likely to remain small as $k \to \infty$.

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Proposition (Computational Complexity of DAG Construction)

For any $\delta \in \left(0, \frac{1}{2}\right)$, the d-regular bipartite expander graphs for levels $0, \ldots, k$ of "expander DAG" can be constructed in:

- deterministic quasi-polynomial time $O(\exp(\Theta(\log(k) \log\log(k))))$,

Remark: Enumerate all d-regular bipartite graphs and test expansion.

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- randomized polylogarithmic time $O(\log(k)\log\log(k))$ with positive success probability (which depends on δ but not k).

Remark: Generate uniform random d-regular bipartite graphs.

Outline

Conclusion

Main Contributions:

- Broadcasting in random DAGs with $d \geq 3$ and majority processing

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- Prove conjecture that for random DAG with odd $d \geq 3$ (or $d = 2$), reconstruction impossible for all processing functions when $\delta \geq \delta_{\text{maj}}$ (or $\delta \geq \delta_{\text{nand}}$).

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- Find polynomial time construction of DAGs with sufficiently large d given some δ such that broadcasting possible.
- Construct DAGs with arbitrary $d \geq 3$ and $\delta < \delta_{\text{maj}}$, or $d = 2$ and $\delta < \delta_{\text{nand}}$, such that broadcasting possible.

Thank You!