Bounds on Permutation Channel Capacity

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Massachusetts Institute of Technology

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Outline

Three Motivations

- **Coding theory:** [DG01], [Mit06], [Met09], [KV15], [KT18], ...

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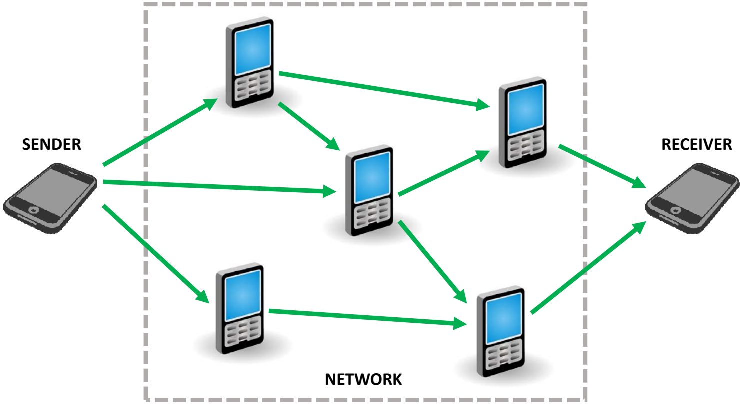

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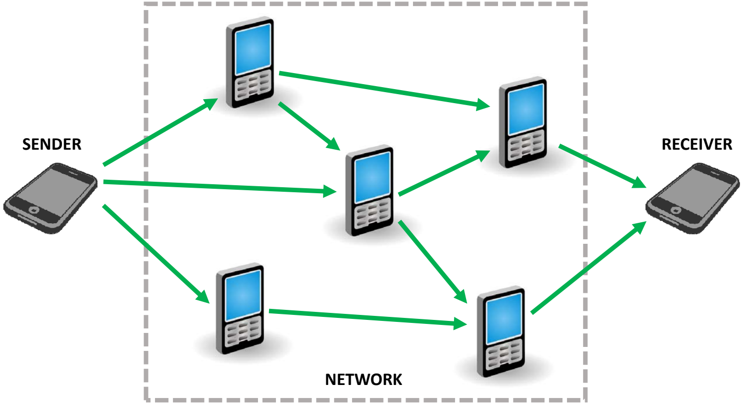

Model communication network as a channel

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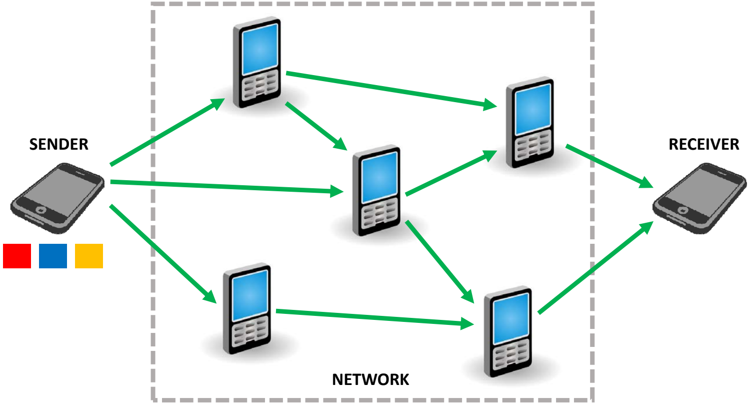

Model communication network as a channel:

- Alphabet symbols = all possible b-bit packets $\Rightarrow$ 2^b input symbols

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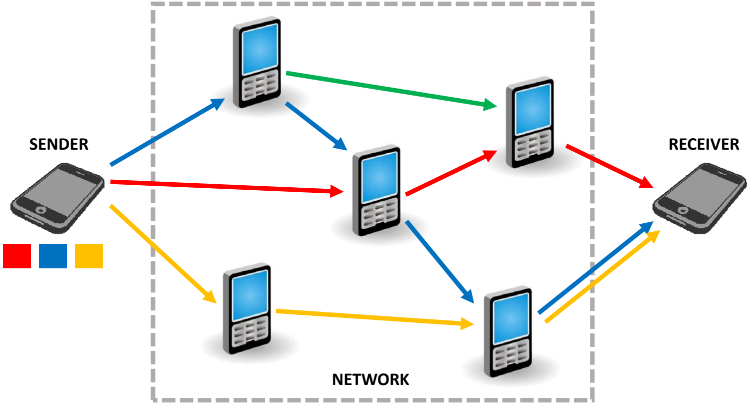

Model communication network as a channel:

- Alphabet symbols = all possible b-bit packets
- Multipath routed network or evolving network topology

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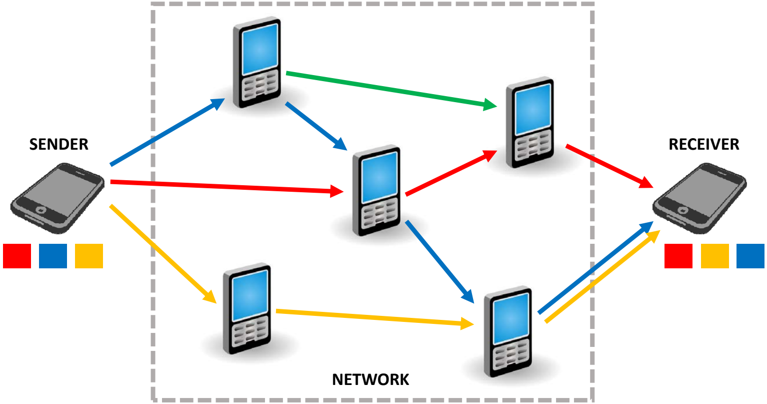

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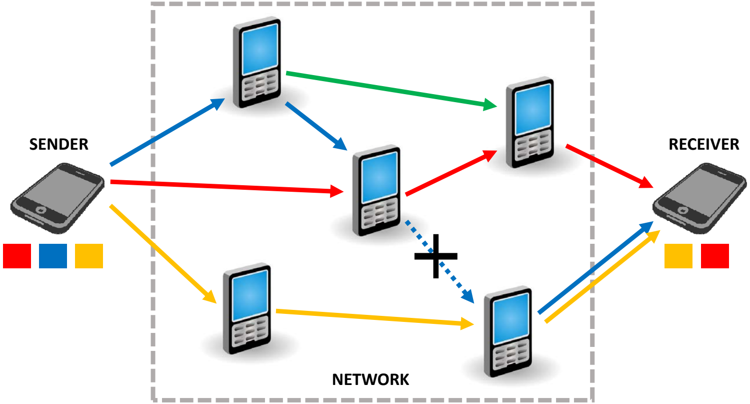

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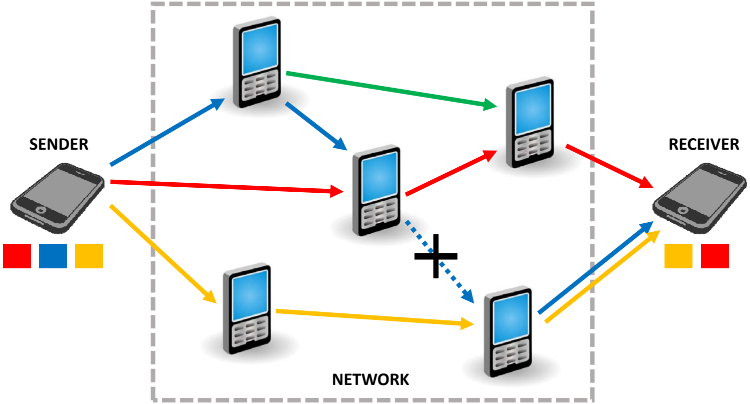
- Packets are impaired (e.g., deletions, substitutions, etc.)



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- Alphabet symbols = all possible b-bit packets
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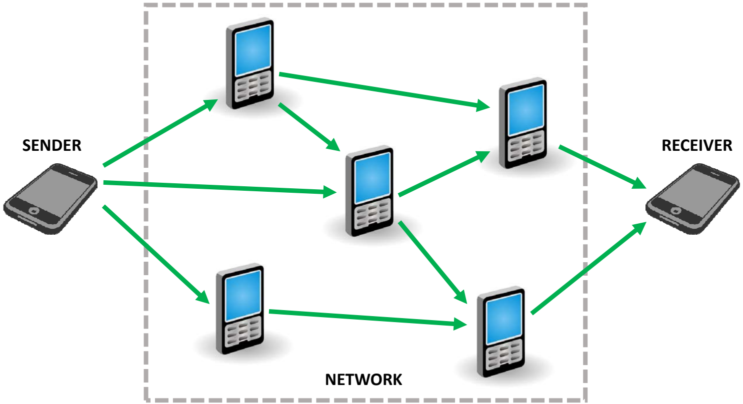
- Packets are impaired $\Rightarrow$ model using channel probabilities

Example: Coding for Random Deletion Network

Consider a communication network where packets can be dropped:



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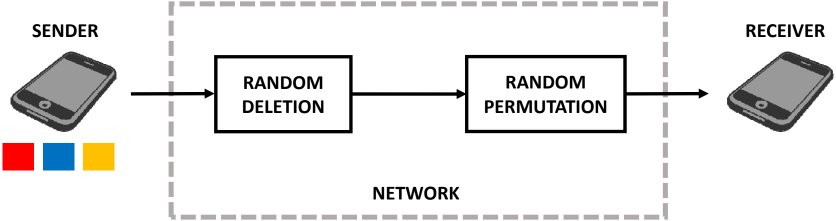

Abstraction:

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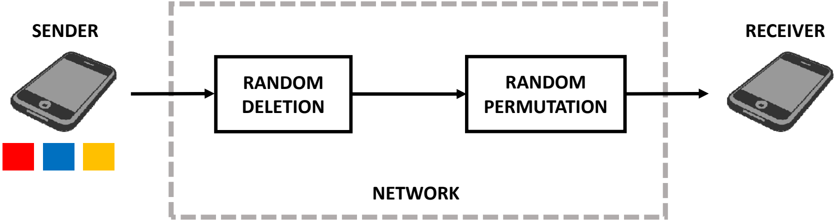

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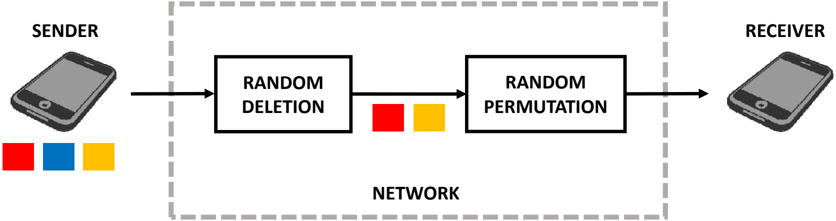

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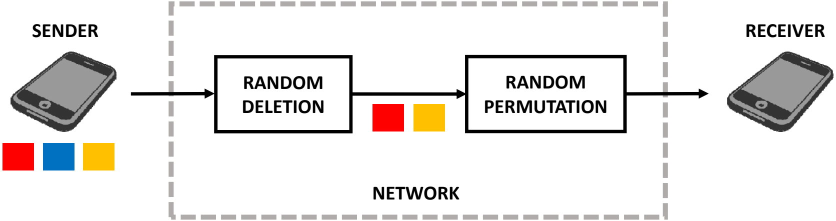

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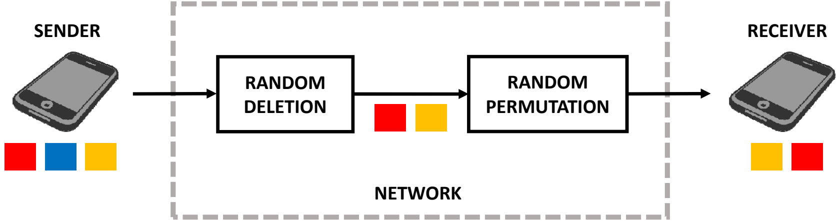

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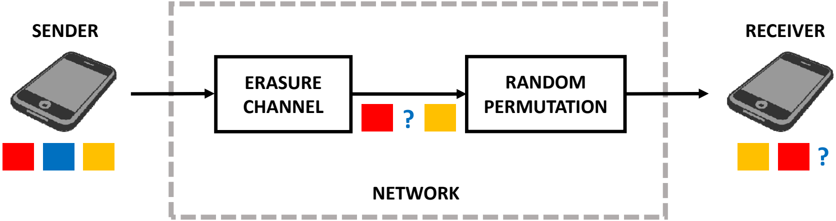

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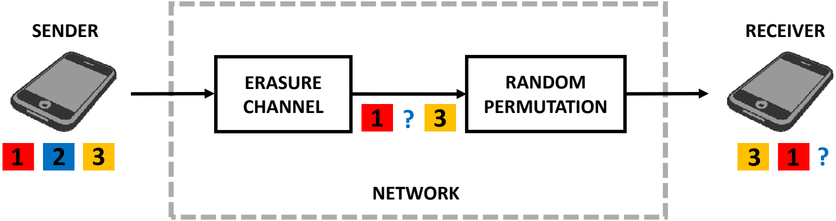

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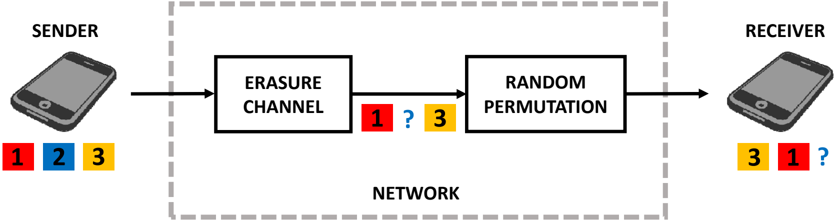

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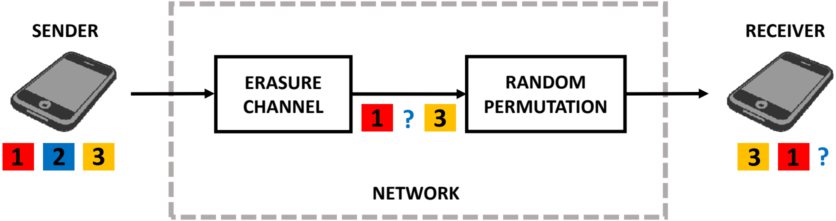

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- More refined coding techniques *simulate* sequence numbers [Mit06], [Met09]

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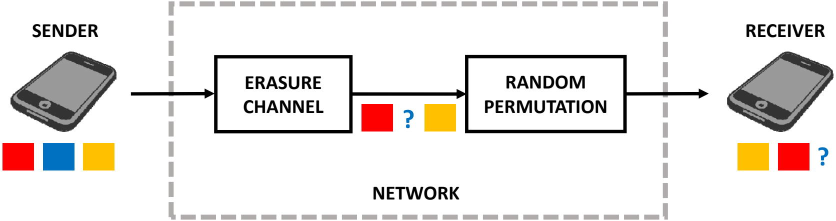

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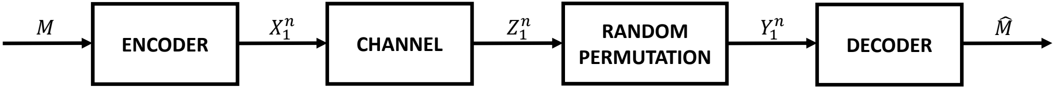
How do you code in such channels without increasing alphabet size?

Permutation Channel Model



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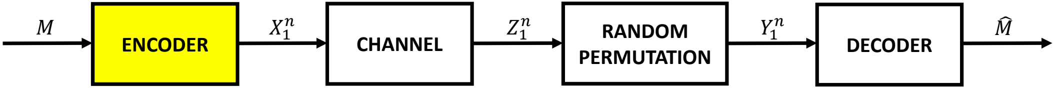

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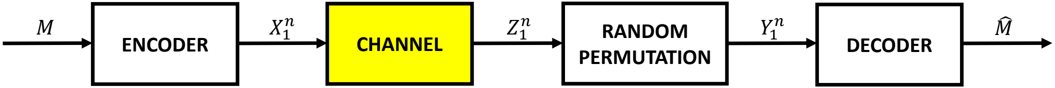

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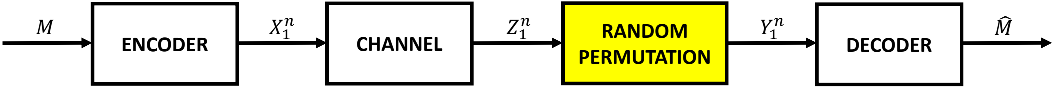

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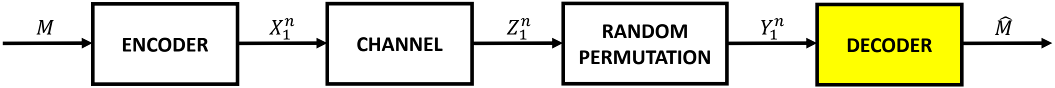

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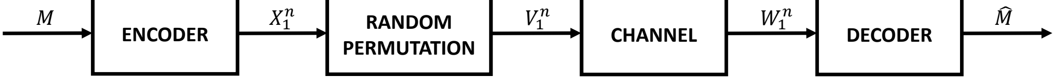
- Randomized decoder $g_n : \mathcal{Y}^n \to \mathcal{M} \cup \{\text{error}\}$ produces estimate $\hat{M} = g_n(Y_1^n)$ at receiver

Permutation Channel Model

What if we analyze the "swapped" model?



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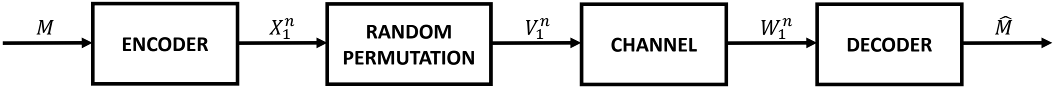

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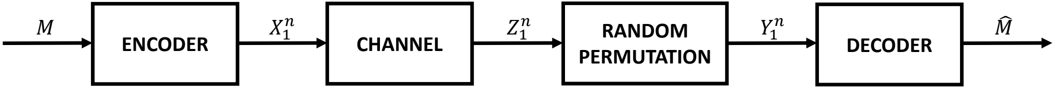

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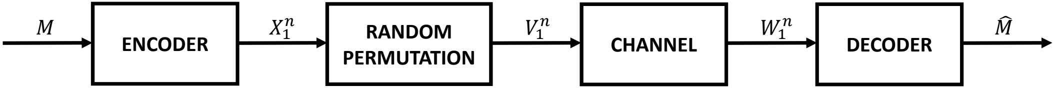

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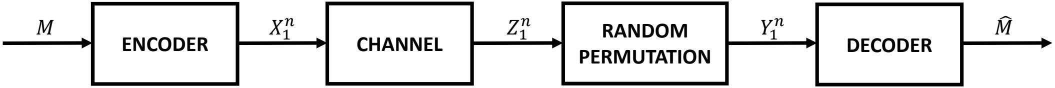

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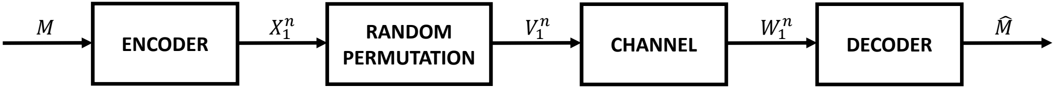

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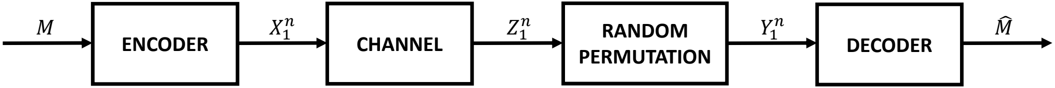

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- Can analyze *either* model!

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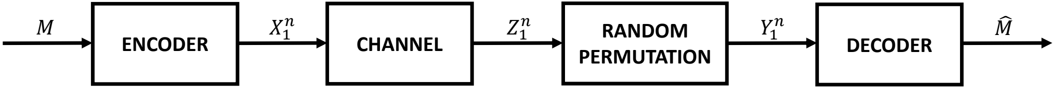

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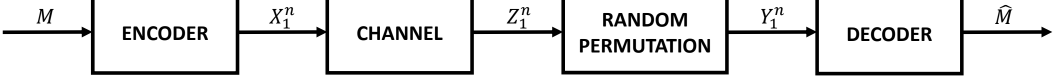

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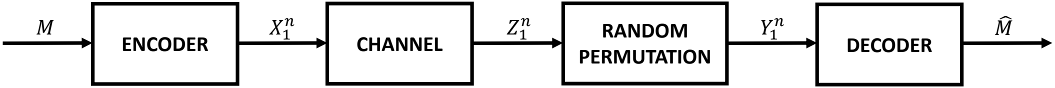

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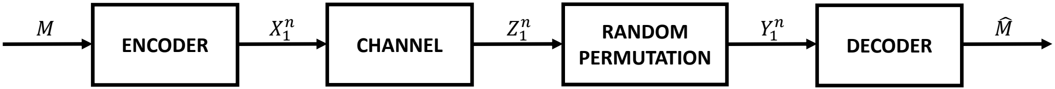
In contrast, in [Mak18], we asked:

What are the fundamental information theoretic limits?

Information Capacity of the Permutation Channel



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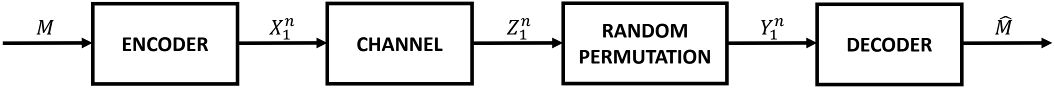

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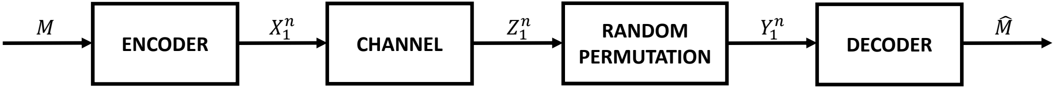

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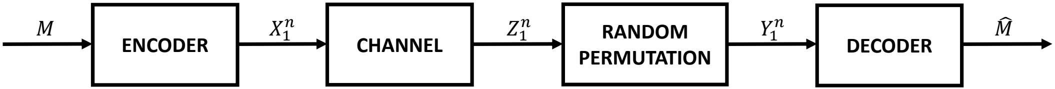

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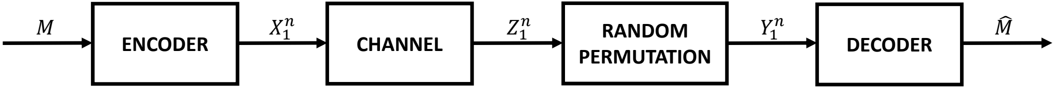
- $|\mathcal{M}| = n^R$ because number of empirical distributions of Y_1^n is $poly(n)$

Information Capacity of the Permutation Channel



- Average probability of error $P_{\text{error}}^n \triangleq \mathbb{P}(M \neq \hat{M})$
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- Rate $R \geq 0$ is achievable $\Leftrightarrow \exists \{(f_n, g_n)\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} P_{\text{error}}^n = 0$

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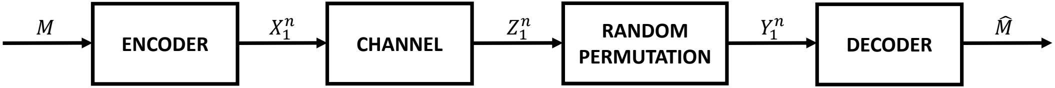

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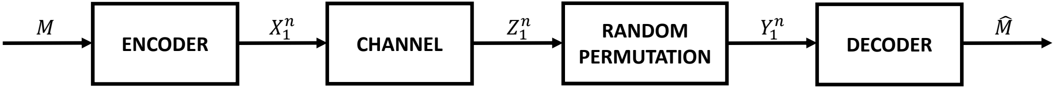

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Main Question

What is the permutation channel capacity of a general $P_{Z|X}$?

Outline

1. Introduction
2. Achievability Bound
   - Coding Scheme
   - Rank Bound
3. Converse Bounds
4. Conclusion

Achievability: Coding Scheme

- Let $r = \text{rank}(P_{Z|X})$ and $k = \lfloor \sqrt{n} \rfloor$

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$$\mathcal{M} \triangleq \left\{ p = (p(x) : x \in \mathcal{X}') \in (\mathbb{Z}_+)^{\mathcal{X}'} : \sum_{x \in \mathcal{X}'} p(x) = k \right\}$$

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- **Randomized Encoder:**

$$\forall p \in \mathcal{M}, \quad f_n(p) = X_1^n \overset{\text{i.i.d.}}{\sim} P_X \quad \text{where} \quad P_X(x) = \begin{cases} \frac{p(x)}{k}, & \text{for } x \in \mathcal{X}' \\ 0, & \text{for } x \in \mathcal{X} \backslash \mathcal{X}' \end{cases}$$

Achievability: Coding Scheme

- Let stochastic matrix $\tilde{P}_{Z|X} \in \mathbb{R}^{r \times |\mathcal{Y}|}$ have rows $\{P_{Z|X}(\cdot|x) : x \in \mathcal{X}'\}$
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$$\forall x \in \mathcal{X}', \quad \hat{p}(x) = \underset{j \in \{0,\dots,k\}}{\arg\min} \left| \sum_{y \in \mathcal{Y}} \hat{P}_{y_1^n}(y) \left[\tilde{P}^{\dagger}_{Z|X}\right]_{y,x} - \frac{j}{k} \right|$$

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  Step 3: Output decoded message

$$g_n(y_1^n) = \begin{cases} \hat{p}, & \text{if } \hat{p} \in \mathcal{M} \\ \text{error}, & \text{otherwise} \end{cases}$$

Achievability: Rank Bound

Theorem (Rank Bound)

For any channel $P_{Z|X}$:

$$C_{\text{perm}}(P_{Z|X}) \geq \frac{\operatorname{rank}(P_{Z|X}) - 1}{2} .$$

Remarks about Coding Scheme:

- Showing $\lim_{n \to \infty} P_{\text{error}}^n = 0$ proves theorem.

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- *Computational complexity*: Decoder has $O(n)$ running time.
- *Probabilistic method*: Good deterministic codes exist.
- *Expurgation*: Achievability bound holds under maximal probability of error criterion.

Outline

Converse: Output Alphabet Bound

Theorem (Output Alphabet Bound)

For any entry-wise *strictly positive* channel $P_{Z|X} > 0$:

$$C_{\mathsf{perm}}(P_{Z|X}) \leq \frac{|\mathcal{Y}| - 1}{2} .$$

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- What if $|\mathcal{X}|$ is much smaller than $|\mathcal{Y}|$?

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- **Want:** Converse bound in terms of input alphabet size.

Converse: Effective Input Alphabet Bound

Theorem (Effective Input Alphabet Bound)

For any entry-wise *strictly positive* channel $P_{Z|X} > 0$:

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where $\mathrm{ext}(P_{Z|X})$ denotes the number of *extreme points* of $\mathrm{conv}\{P_{Z|X}(\cdot|x) : x \in \mathcal{X}\}$.

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Remarks:

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- For any *general* channel $P_{Z|X}$, $C_{\text{perm}}(P_{Z|X}) \leq \min\{\text{ext}(P_{Z|X}), |\mathcal{Y}|\} - 1$.

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- **How do we prove above theorem?**

Proof Idea: Degradation by Symmetric Channels

Definition (Degradation/Blackwell Order [Bla51], [She51], [Ste51], [Cov72], [Ber73])

Given channels $P_{Z_1|X}$ and $P_{Z_2|X}$ with common input alphabet $\mathcal{X}$, $P_{Z_2|X}$ is a degraded version of $P_{Z_1|X}$ if $P_{Z_2|X} = P_{Z_1|X}P_{Z_2|Z_1}$ for some channel $P_{Z_2|Z_1}$.

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Definition (q-ary Symmetric Channel)

A q-ary symmetric channel, denoted q-SC(δ), with total crossover probability $\delta \in [0, 1]$ and alphabet $\mathcal{X}$ where $|\mathcal{X}| = q$, is given by the doubly stochastic matrix:

$$W_\delta \triangleq \begin{bmatrix} 1-\delta & \frac{\delta}{q-1} & \cdots & \frac{\delta}{q-1} \\ \frac{\delta}{q-1} & 1-\delta & \cdots & \frac{\delta}{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta}{q-1} & \frac{\delta}{q-1} & \cdots & 1-\delta \end{bmatrix}.$$

Proof Idea: Degradation by Symmetric Channels

Proposition (Degradation by Symmetric Channels)

Given channel $P_{Z|X}$ with $\nu = \min_{x \in \mathcal{X}, y \in \mathcal{Y}} P_{Z|X}(y|x)$, if we have:

$$0 \leq \delta \leq \frac{\nu}{1 - \nu + \frac{\nu}{q-1}},$$

then $P_{Z|X}$ is a degraded version of q-SC(δ).

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- Prop follows from computing extremal δ such that $W_\delta^{-1} P_{Z|X}$ is row stochastic.
- Many other applications in information theory and statistics [MP18], [MOS13].

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- Prop + "swapped" model + *tensorization* of degradation $\Rightarrow$ $I(X_1^n; Y_1^n) \leq I(X_1^n; \tilde{Y}_1^n)$, where Y_1^n and $\tilde{Y}_1^n$ are outputs of permutation channels with $P_{Z|X}$ and q-SC(δ).

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- *Convexity* of KL divergence $\Rightarrow$ Reduce $|\mathcal{X}|$ to ext$(P_{Z|X})$.

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- *Fano argument* of output alphabet bound $\Rightarrow$ effective input alphabet bound.

Outline

Strictly Positive and "Full Rank" Channels

Achievability and converse bounds yield:

Theorem (Strictly Positive and "Full Rank" Channels)

For any entry-wise *strictly positive* channel $P_{Z|X} > 0$ that is *"full rank"* in the sense that $r \triangleq \text{rank}(P_{Z|X}) = \min\{\text{ext}(P_{Z|X}), |\mathcal{Y}|\}$:

$$C_{\text{perm}}(P_{Z|X}) = \frac{r-1}{2} .$$

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Example [Mak18]**:** C_{perm} of non-trivial binary symmetric channel is $\frac{1}{2}$.

Conclusion

Main Result:

For any entry-wise *strictly positive* channel $P_{Z|X} > 0$:

$$\frac{\text{rank}(P_{Z|X}) - 1}{2} \leq C_{\text{perm}}(P_{Z|X}) \leq \frac{\min\{\text{ext}(P_{Z|X}), |\mathcal{Y}|\} - 1}{2}.$$

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For any entry-wise *strictly positive* channel $P_{Z|X} > 0$:

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Future Direction:

Characterize C_{perm} of all entry-wise strictly positive channels, and more generally, all channels.

Thank You!