Reconstruction on 2D Regular Grids

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Outline

Motivation: Information Propagation in Networks

social networks

communication networks

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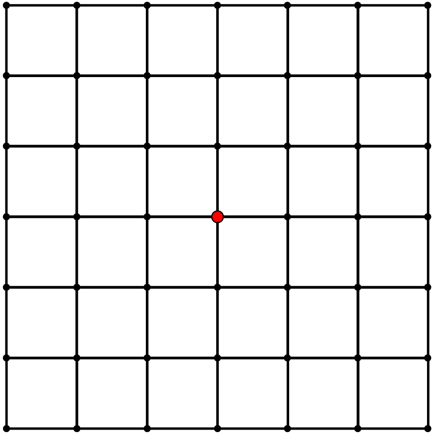
- How does information propagate through such large networks over time?



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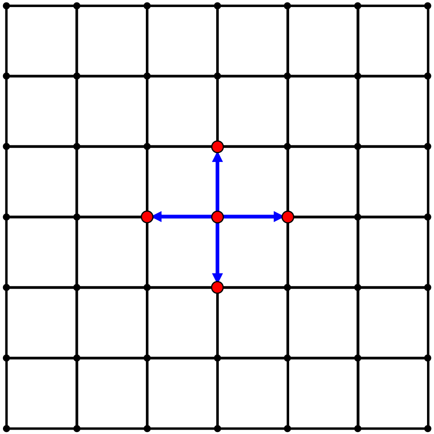

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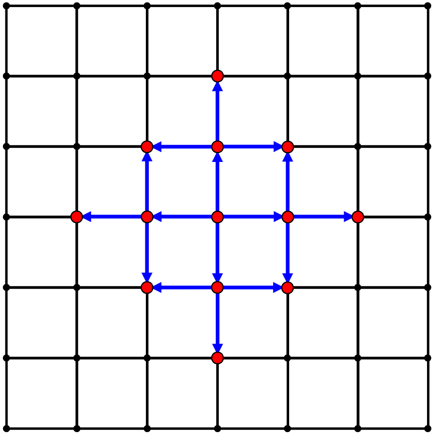

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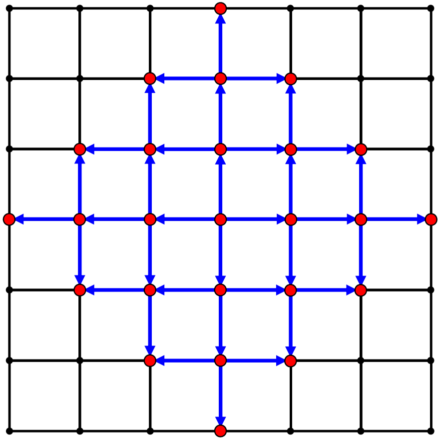

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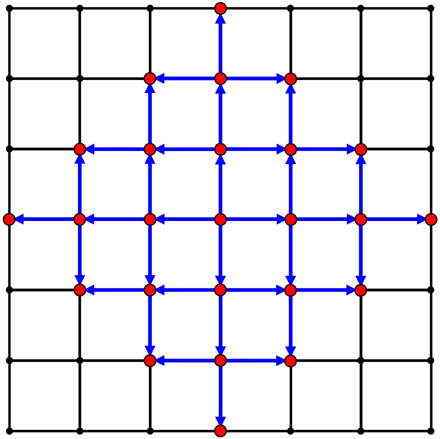
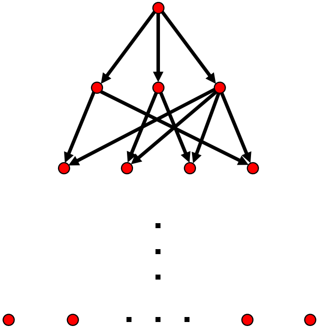
- How does information propagate through such large networks over time?
- Can we invent *processing functions* so that far boundary has information about source bit?



Formal Model: Bounded Indegree DAGs

- Fix infinite directed acyclic graph (DAG) with single source node



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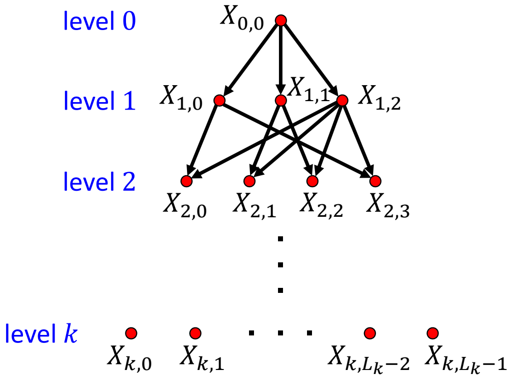
- $X_{k,j} \in \{0, 1\}$ – node random variable at jth position in level k



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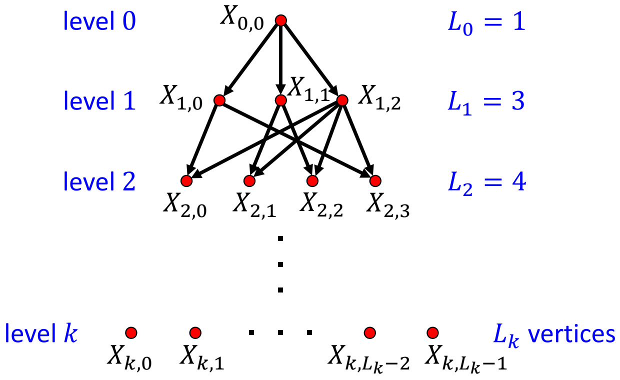

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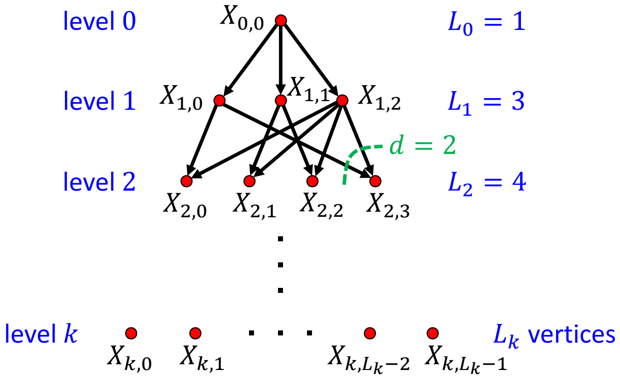

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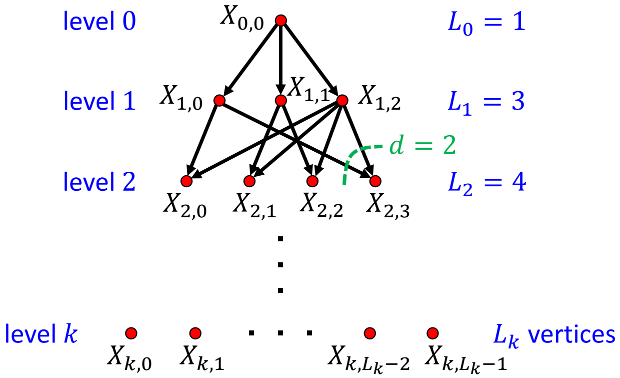

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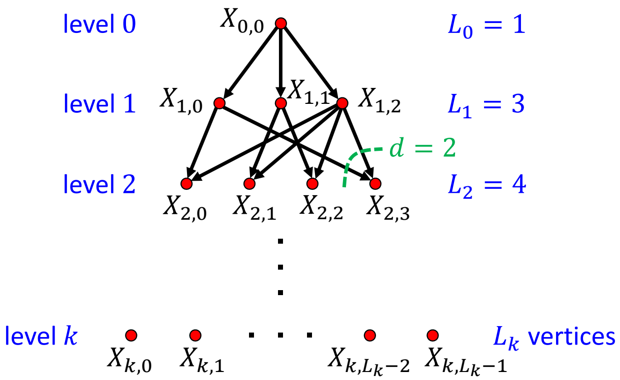

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- Nodes combine inputs with Boolean processing functions
- This defines joint distribution of $\{X_{k,j}\}$

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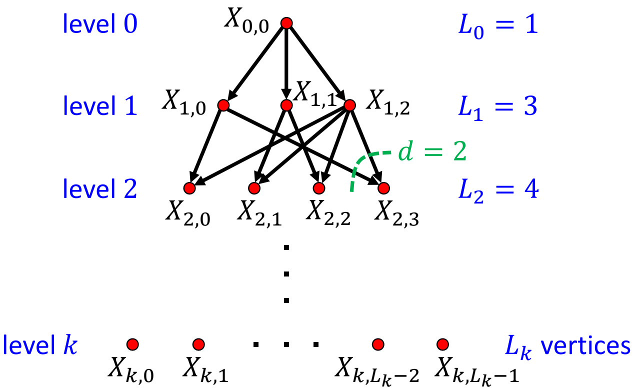
- Let $X_k \triangleq (X_{k,0}, \ldots, X_{k,L_k-1})$
- **Question:** Can we decode X_0 from X_k as $k \to \infty$?



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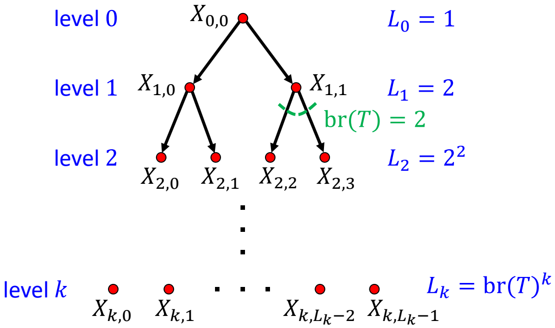
For which graph topologies, noise levels δ, and Boolean processing functions is reconstruction possible?

Background: Reconstruction on Trees

- Suppose DAG is tree T with identity processing and branching number $\mathrm{br}(T)$



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Phase Transition for Trees: [Kesten-Stigum 1966, Bleher-Ruiz-Zagrebnov 1995, Evans *et al.* 2000]

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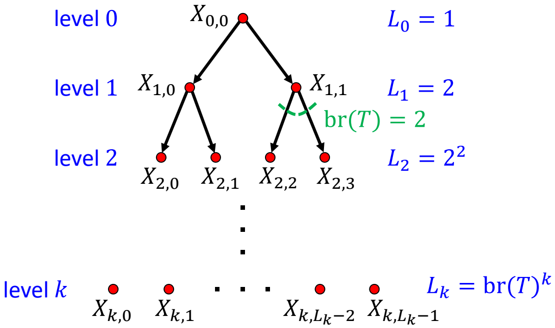

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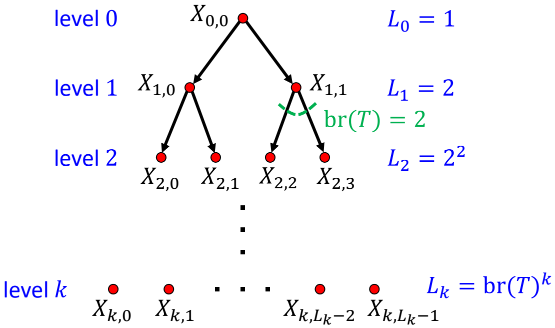

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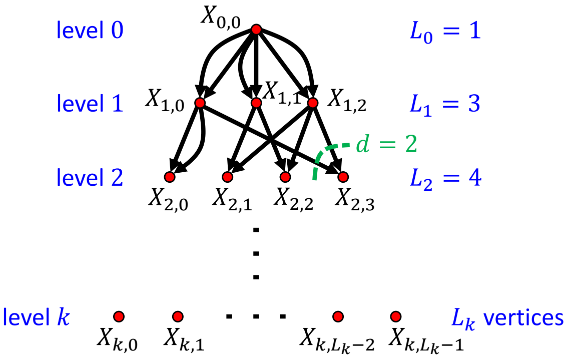

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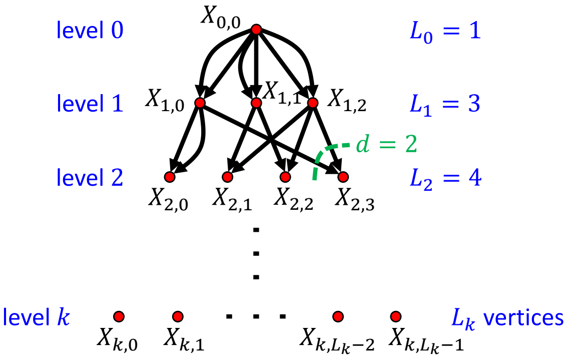

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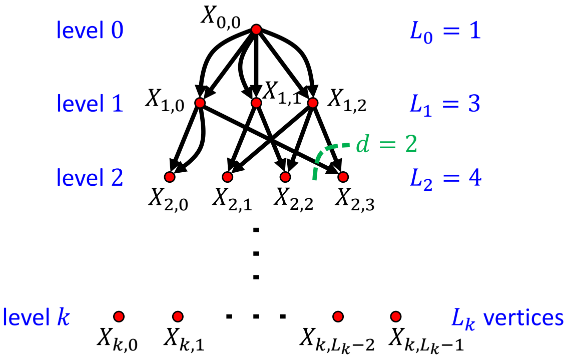

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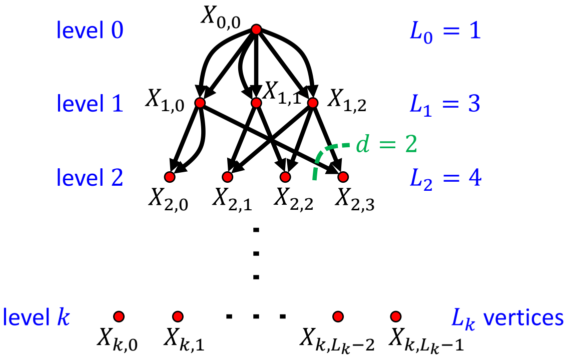

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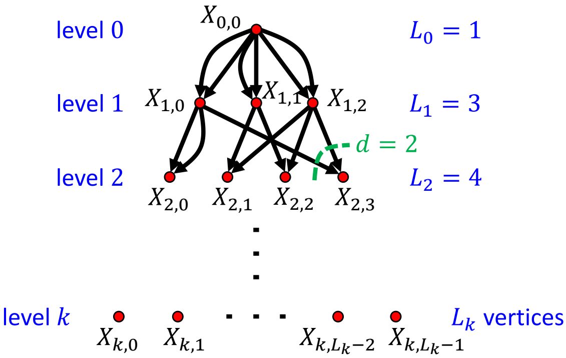

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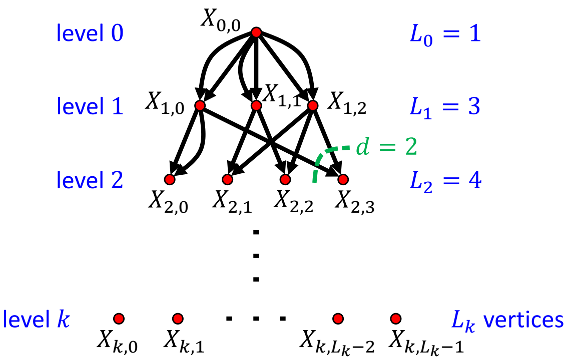

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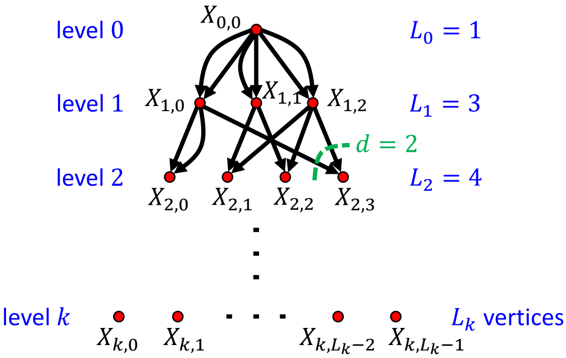

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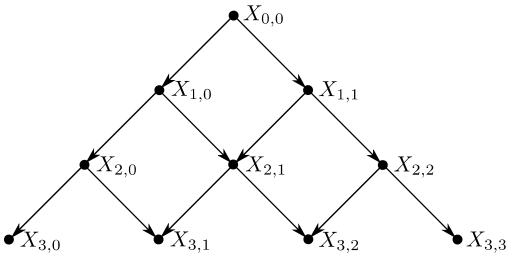
2D Regular Grid Model



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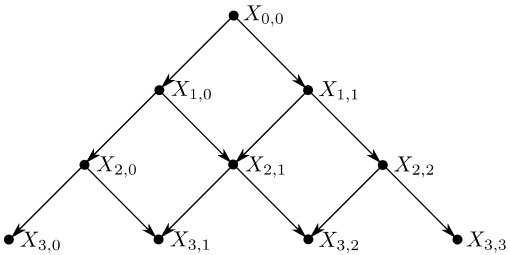

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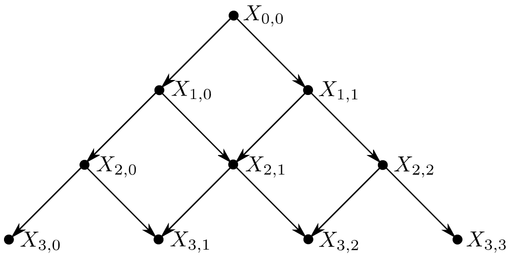

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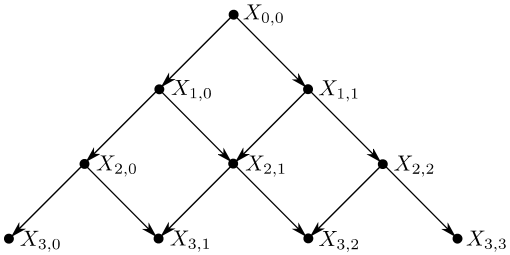
Conjecture: For all $\delta \in \left(0, \frac{1}{2}\right)$ and all choices of processing functions, reconstruction impossible: $\lim_{k \to \infty} P_{\mathsf{ML}}^{(k)} = \frac{1}{2}$

- **Motivation:** "Positive rates conjecture" on ergodicity of simple 1D probabilistic cellular automata (e.g., [Gray 2001])

Impossibility Result for AND Processing



- Common processing function = AND gate

x	y	$x \wedge y$
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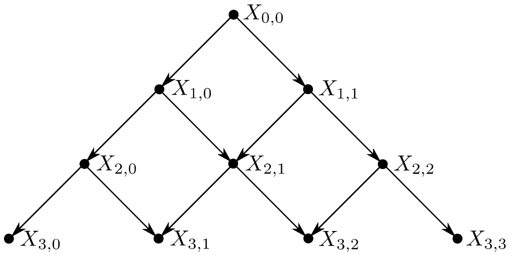

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Theorem (Reconstruction with AND Gates)

Reconstruction impossible on 2D regular grid with AND processing functions for all $\delta \in \left(0, \frac{1}{2}\right)$

Proof Sketch: AND Processing

- **Step 1: Monotone Markovian coupling**
  - Layers $\{X_k = (X_{k,0}, \ldots, X_{k,k})\}$ form a Markov chain

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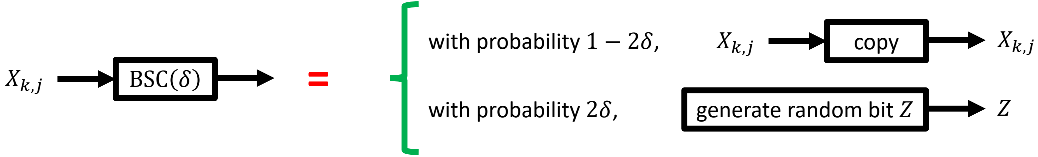
  - Representation of BSC(δ): (Z is Bernoulli$\left(\frac{1}{2}\right)$)



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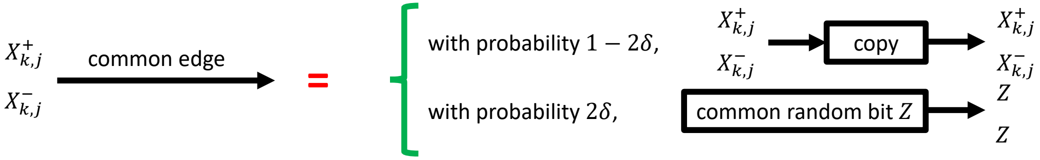
  - Couple $\{X_k^+\}$ and $\{X_k^-\}$ to run on common 2D regular grid:



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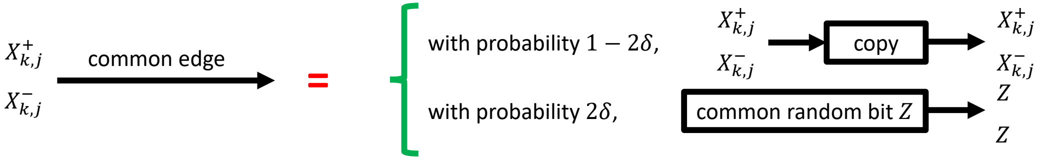

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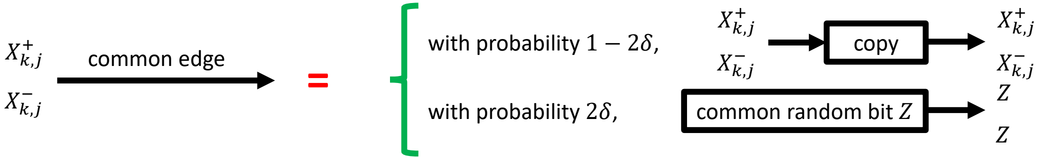

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  - Define coupled grid variables $Y_{k,j} = (X_{k,j}^-, X_{k,j}^+) \in \{0_c, 1_u, 1_c\}$ with source $Y_{0,0} = 1_u$, where $0_c = (0,0)$, $1_u = (0,1)$, $1_c = (1,1)$

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- **Step 2: Reduction to coupled grid**
  - Define coupled grid variables $Y_{k,j} = (X^-_{k,j}, X^+_{k,j}) \in \{0_c, 1_u, 1_c\}$ with source $Y_{0,0} = 1_u$

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  - Maximal coupling characterization of *total variation distance* $\|\cdot\|_{\mathsf{TV}}$:

$$\left\| P_{X_k}^+ - P_{X_k}^- \right\|_{\mathsf{TV}} \leq \mathbb{P}\left(X_k^+ \neq X_k^-\right) = 1 - \mathbb{P}(\forall j,\ Y_{k,j} \neq 1_\mathrm{u})$$

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  - Using Step 1, $\lim_{k\to\infty} \left\| P_{X_k}^+ - P_{X_k}^- \right\|_{\mathsf{TV}} \leq 1 - \mathbb{P}(\exists k, \forall j,\ Y_{k,j} \neq 1_u)$

Proof Sketch: AND Processing

- **Step 2: Reduction to coupled grid**
  - Define coupled grid variables $Y_{k,j} = (X_{k,j}^-, X_{k,j}^+) \in \{0_\text{c}, 1_\text{u}, 1_\text{c}\}$ with source $Y_{0,0} = 1_\text{u}$
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  - By Le Cam's relation,

$$\lim_{k\to\infty} P_{\text{ML}}^{(k)} = \frac{1}{2}\left(1 - \lim_{k\to\infty} \left\| P_{X_k}^+ - P_{X_k}^- \right\|_{\text{TV}}\right) \geq \frac{1}{2}\,\mathbb{P}(\exists k,\ \forall j,\ Y_{k,j} \neq 1_\text{u})$$

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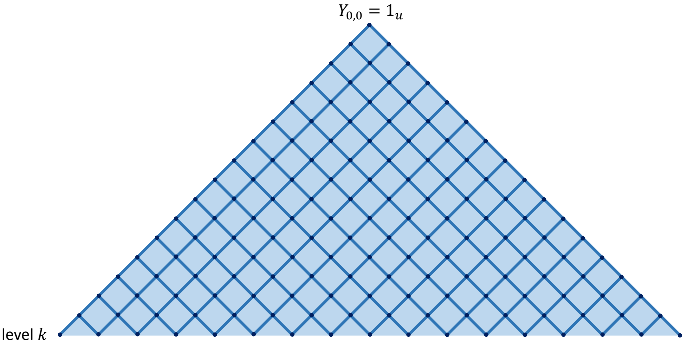
$$\lim_{k\to\infty} P_{ML}^{(k)} = \frac{1}{2}\left(1 - \lim_{k\to\infty} \left\| P_{X_k}^+ - P_{X_k}^- \right\|_{TV}\right) \geq \frac{1}{2}\mathbb{P}(\exists k,\ \forall j,\ Y_{k,j} \neq 1_u)$$

  - To show $\lim_{k\to\infty} P_{ML}^{(k)} = \frac{1}{2}$, it *suffices to prove* that $\mathbb{P}(\exists k,\ \forall j,\ Y_{k,j} \neq 1_u) = 1$

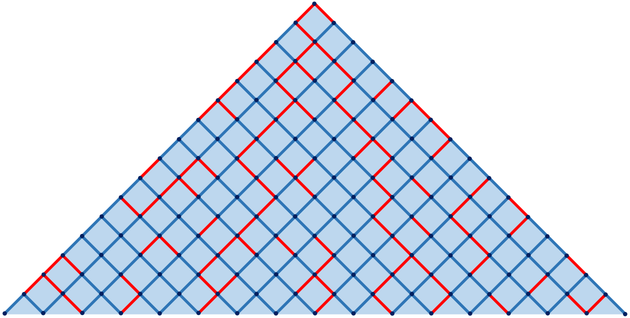
Proof Sketch: AND Processing

- **Step 3:** **Bond percolation**



Proof Sketch: AND Processing

- **Step 3: Bond percolation**
  - Independently keep each edge **open** with probability $p \in [0,1]$, and *closed* with probability $1-p$



Proof Sketch: AND Processing

- **Step 3: Bond percolation**
  - Independently keep each edge **open** with probability $p \in [0,1]$, and *closed* with probability $1-p$
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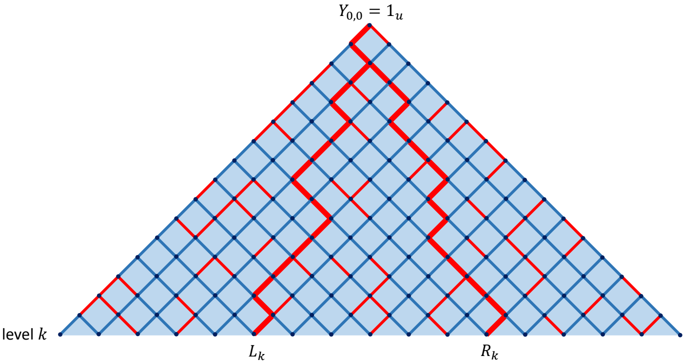

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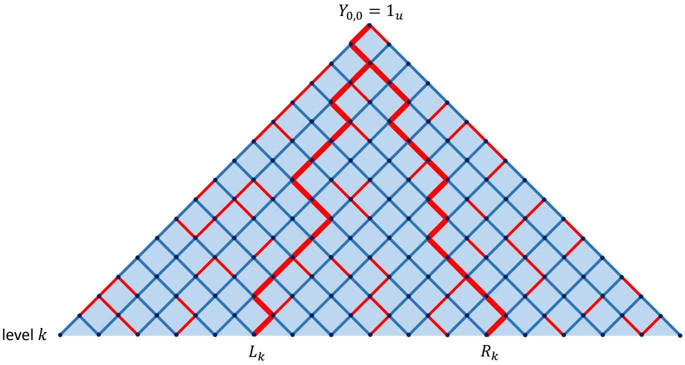

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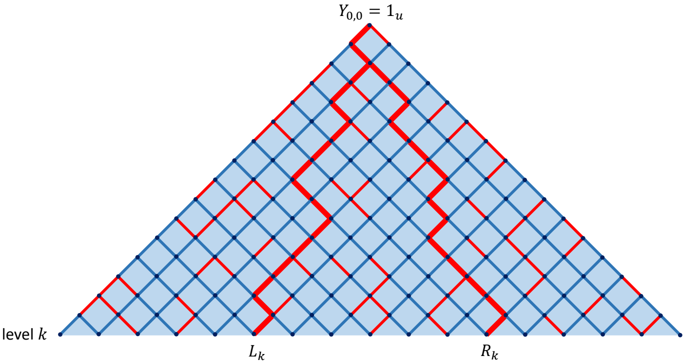

- **Phase transition [Durrett 1984]:** There is a critical threshold $\delta_{\text{perc}} \in \left(\frac{1}{2}, 1\right)$ such that:
  - If $p < \delta_{\text{perc}}$, then $\mathbb{P}(\Omega_\infty) = 0$

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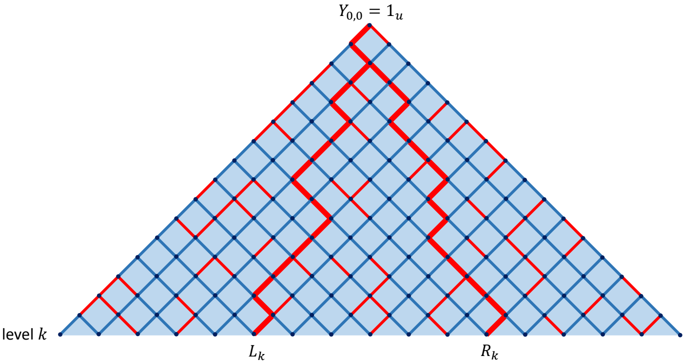

- **Phase transition** [Durrett 1984]**:** There is a critical threshold $\delta_{\text{perc}} \in \left(\frac{1}{2}, 1\right)$ such that:
  - If $p < \delta_{\text{perc}}$, then $\mathbb{P}(\Omega_\infty) = 0$
  - If $p > \delta_{\text{perc}}$, then $\mathbb{P}(\Omega_\infty) > 0$ and $\mathbb{P}\left(\lim_{k\to\infty} \frac{R_k - L_k}{k} > 0 \,\middle|\, \Omega_\infty\right) = 1$

Proof Sketch: AND Processing

- **Step 2:** Suffices to prove $\mathbb{P}(\exists k, \forall j, Y_{k,j} \neq 1_{\mathsf{u}}) = 1$

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  - Bond percolation: Edge open $\Leftrightarrow$ BSC(δ) copies input, i.e., $p = 1 - 2\delta$

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Proof Sketch: AND Processing

- **Step 2:** Suffices to prove $\mathbb{P}(\exists k, \forall j, Y_{k,j} \neq 1_{\mathsf{u}}) = 1$ ✓
- **Step 4: Case I (Noise level satisfies $p = 1 - 2\delta < \delta_{\mathsf{perc}}$)**
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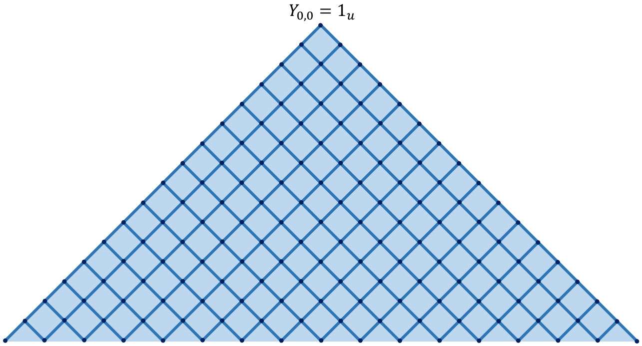
  - Step 3 $\Rightarrow$ There is no infinite open path from $Y_{0,0}$ almost surely
  - Markovian coupling of Step 1 $\Rightarrow$ 1_{u}'s only travel along open edges
  - Hence, there exists level k such that $Y_{k,j} \neq 1_{\mathsf{u}}$ for all j

Proof Sketch: AND Processing

- **Step 2:** Suffices to prove $\mathbb{P}(\exists r, \forall j, Y_{r,j} \neq 1_{\text{u}}) = 1$
- **Step 5:** **Case II (Noise level satisfies $p = 1 - \delta > \delta_{\text{perc}}$)**
  - Bond percolation: Edge open $\Leftrightarrow$ BSC(δ) copies or generates random bit $= 0$, i.e., $p = 1 - \delta$



Proof Sketch: AND Processing

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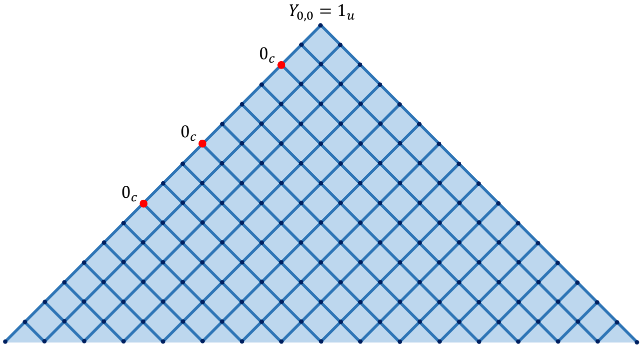
  - Bond percolation: Edge open $\Leftrightarrow$ BSC(δ) copies or generates random bit $= 0$, i.e., $p = 1 - \delta$
  - Boundary BSC(δ)'s generate random bits $= 0$ independently with probability δ



Proof Sketch: AND Processing

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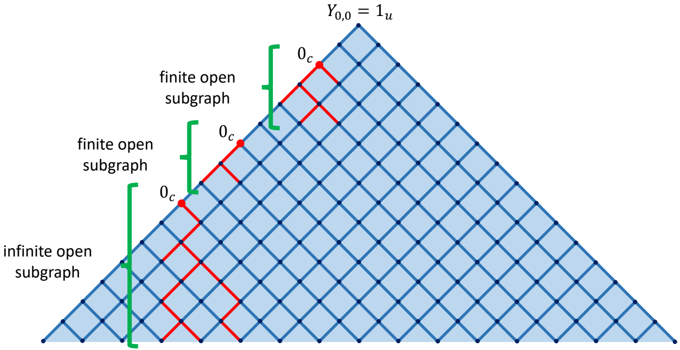
  - Step 3 $\Rightarrow$ 0_c nodes subtend infinite *open subgraph* with probability $\mathbb{P}(\Omega_\infty) > 0$



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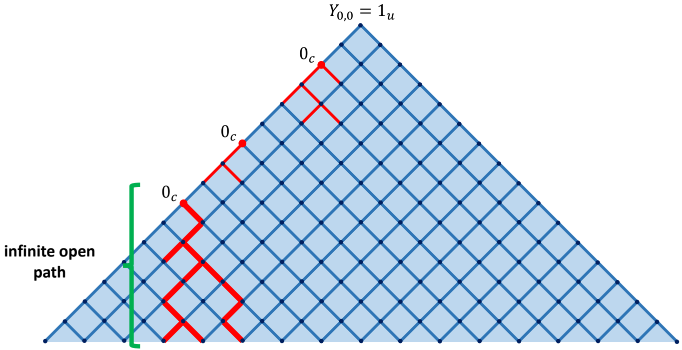
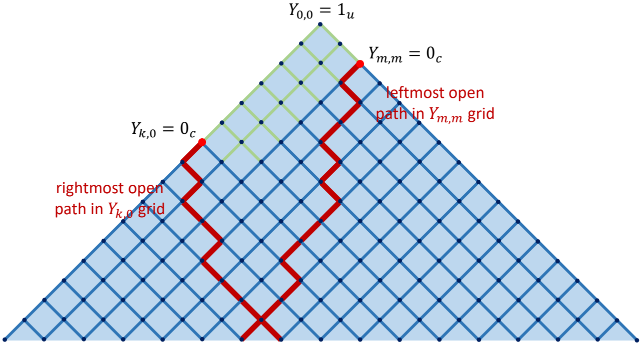
  - *Borel-Cantelli* $\Rightarrow$ There exists 0_c boundary node with infinite open path almost surely



Proof Sketch: AND Processing

- **Step 2:** Suffices to prove $\mathbb{P}(\exists r, \forall j, \; Y_{r,j} \neq 1_\mathrm{u}) = 1$
- **Step 5:** **Case II (Noise level satisfies $p = 1 - \delta > \delta_{\mathrm{perc}}$)**
  - For some levels k and m, $Y_{k,0} = Y_{m,m} = 0_\mathrm{c}$ and both nodes have infinite open paths almost surely



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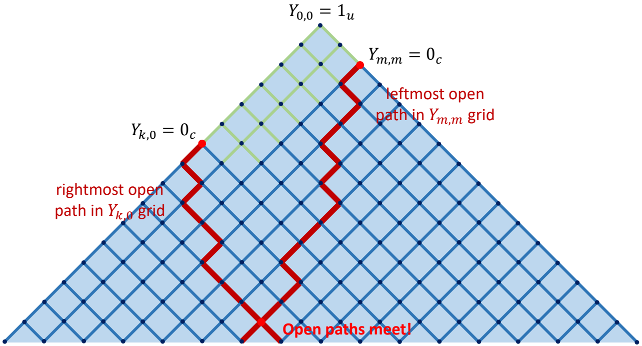
  - Step 3 $\Rightarrow$ Open paths from $Y_{k,0}$ and $Y_{m,m}$ meet



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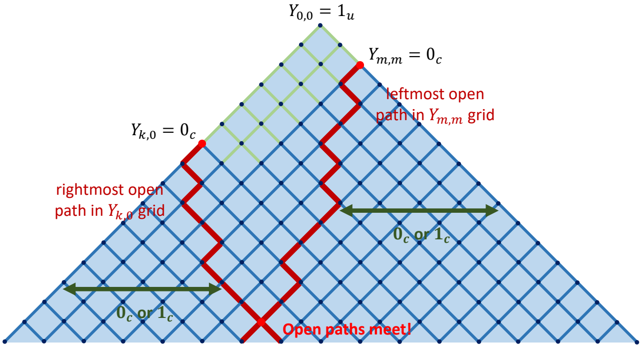
  - Markovian coupling of Step 1 $\Rightarrow$ Nodes enclosed by open paths are 0_{c} or 1_{c}



Proof Sketch: AND Processing

- **Step 2:** Suffices to prove $\mathbb{P}(\exists r, \forall j, Y_{r,j} \neq 1_\mathrm{u}) = 1$ ✓
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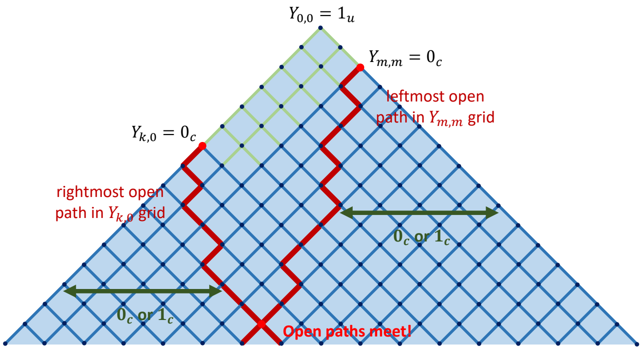
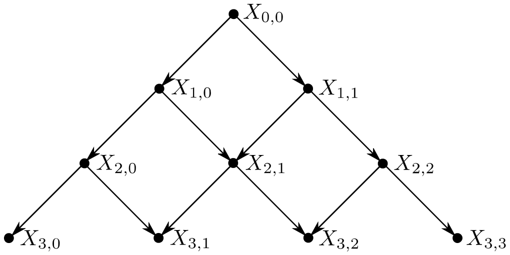


Proof Sketch: AND Processing

- **Step 1:** Monotone Markovian coupling
- **Step 2:** Reduction to coupled grid
- **Step 3:** Bond percolation
- **Step 4:** Case I - Noise level $\delta > (1 - \delta_{\text{perc}})/2$
- **Step 5:** Case II - Noise level $\delta < 1 - \delta_{\text{perc}}$

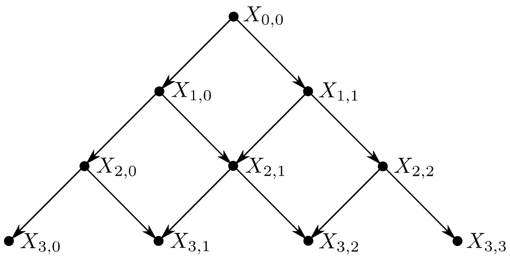
Impossibility Result for XOR Processing



- Common processing function = XOR gate

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

Impossibility Result for XOR Processing



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Theorem (Reconstruction with XOR Gates)

Reconstruction impossible on 2D regular grid with XOR processing functions for all $\delta \in \left(0, \frac{1}{2}\right)$

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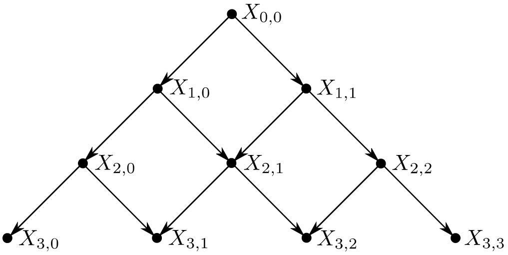

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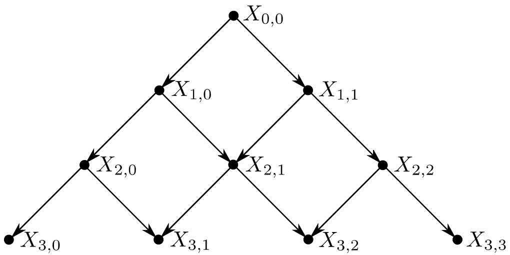
- Proof uses bit-wise ML decoding of linear codes

Impossibility Result for NAND Processing



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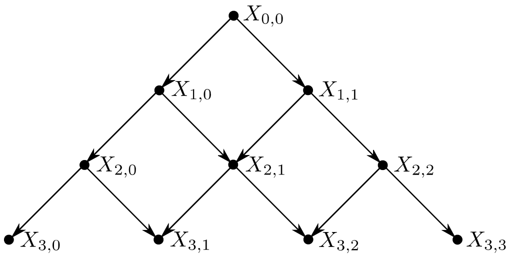

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- Fix known vector b and matrix $A(\delta)$ for any noise level $\delta \in \left(0, \frac{1}{2}\right)$

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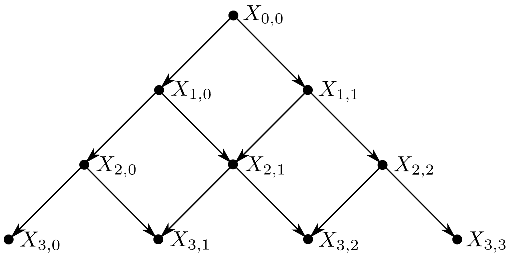

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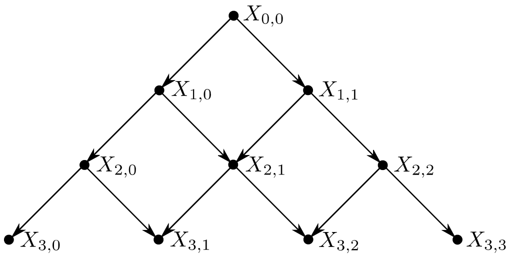
For all $\delta \in \left(0, \frac{1}{2}\right)$, if linear programming (LP) feasibility problem, $A(\delta)\, x \geq b$, has solution $x = x^*(\delta)$, then reconstruction impossible on 2D regular grid with NAND processing functions

Impossibility Result for NAND Processing



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- Proof uses martingale argument and LP solution generates desired martingale

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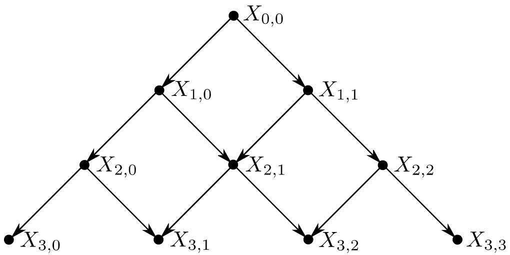

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For all $\delta \in \left(0, \frac{1}{2}\right)$, if linear programming (LP) feasibility problem, $A(\delta)\, x \geq b$, has solution $x = x^*(\delta)$, then reconstruction impossible on 2D regular grid with NAND processing functions

- Proof uses martingale argument and LP solution generates desired martingale
- LPs computationally solved for numerous values of $\delta \in \left(0, \frac{1}{2}\right)$

Outline

Conclusion

Main Contribution:

- Reconstruction impossible in 2D regular grids with *symmetric* processing functions

Conclusion

Main Contribution:

- Reconstruction impossible in 2D regular grids with *symmetric* processing functions

Future Direction:

- Conjecture: For 2D regular grids, reconstruction impossible for *all* processing functions

Thank You!