Reconstruction on 2D Regular Grids

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Motivation: Information Propagation in Networks

social networks

communication networks

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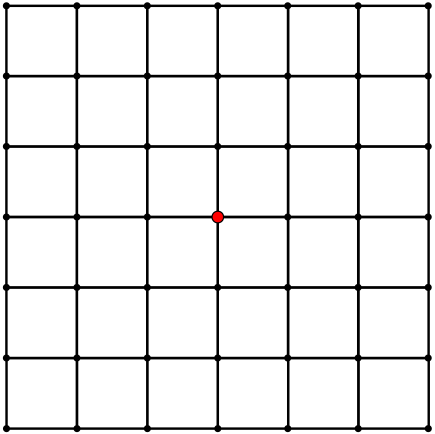
- How does information propagate through such large networks over time?



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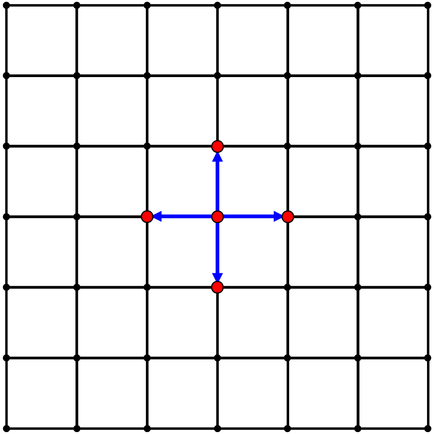

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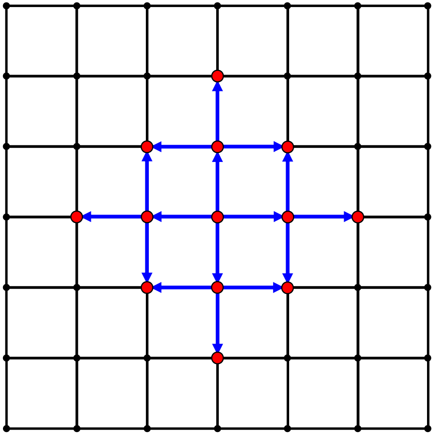

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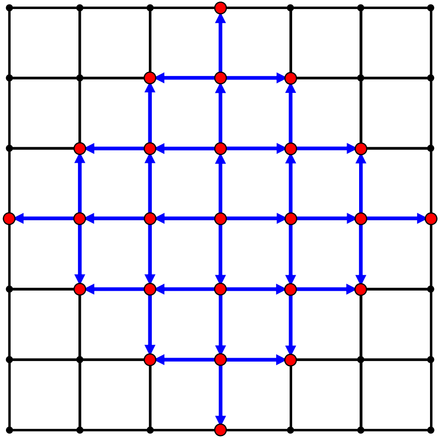
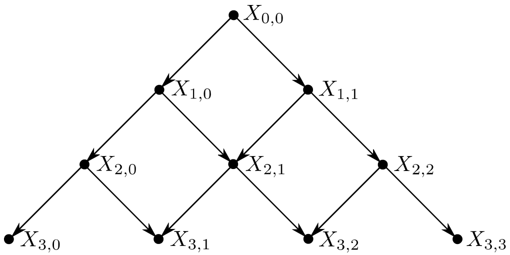
- How does information propagate through such large networks over time?
- Can we invent *processing functions* so that far boundary has information about source bit?



2D Regular Grid Model



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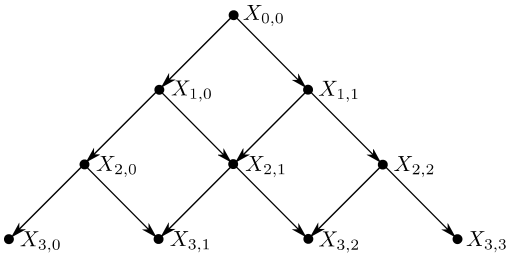

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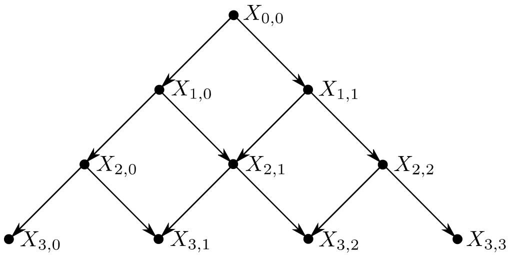

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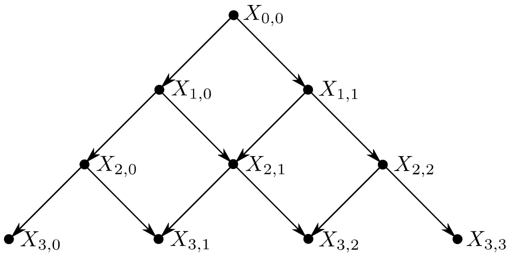

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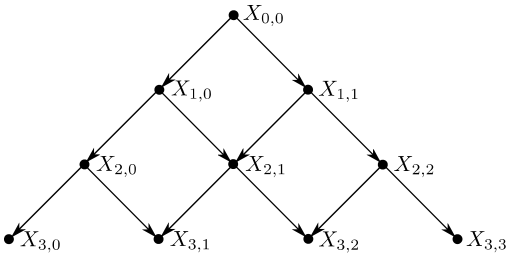

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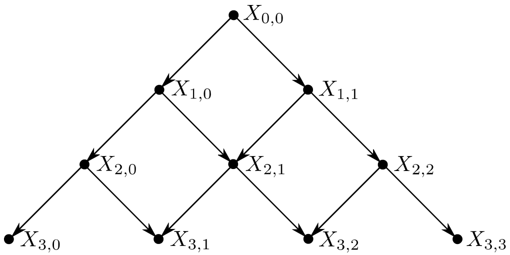

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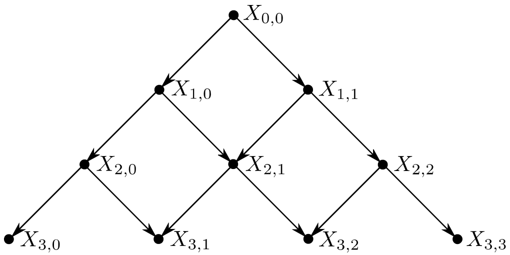

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- **Motivation:** "Positive rates conjecture" on ergodicity of simple 1D probabilistic cellular automata (e.g., [Gray 2001])

Main Results

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- XOR proof exploits bit-wise maximum likelihood decoding of linear codes
- NAND proof uses martingale argument and LP solution generates desired martingale

Thank You!