Comparison of Channels: Criteria for Domination by a Symmetric Channel

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Outline

Preliminaries

- A channel is a set of conditional distributions $W_{Y|X}$ that is represented by a row stochastic matrix $W \in \mathbb{R}_{\text{sto}}^{q \times r}$.

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- Recall that KL divergence is defined as:

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Definition (Less Noisy Preorder [KM77])

$W \in \mathbb{R}_{\mathsf{sto}}^{q \times r}$ is less noisy than $V \in \mathbb{R}_{\mathsf{sto}}^{q \times s}$, denoted $W \succeq_{\mathsf{ln}} V$, iff:

$$\forall P_X, Q_X \in \mathcal{P}_q, \;\; D(P_X W||Q_X W) \geq D(P_X V||Q_X V).$$

Guiding Question

Definition (q-ary Symmetric Channel)

The q-ary symmetric channel is defined as:

$$W_\delta \triangleq \begin{bmatrix} 1-\delta & \frac{\delta}{q-1} & \cdots & \frac{\delta}{q-1} \\ \frac{\delta}{q-1} & 1-\delta & \cdots & \frac{\delta}{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta}{q-1} & \frac{\delta}{q-1} & \cdots & 1-\delta \end{bmatrix} \in \mathbb{R}_{\mathsf{sto}}^{q \times q}$$

where $\delta \in [0,1]$ is the total crossover probability.

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Remark: For every channel $V \in \mathbb{R}_{\mathsf{sto}}^{q \times s}$, $W_0 \succeq_{\mathsf{ln}} V$ and $V \succeq_{\mathsf{ln}} W_{(q-1)/q}$.

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What is the q-ary symmetric channel with the largest $\delta \in \left[0, \frac{q-1}{q}\right]$ that is less noisy than a given channel V?

Motivation: Strong Data Processing Inequality

Data Processing Inequality: For any channel $V \in \mathbb{R}_{\text{sto}}^{q \times s}$,

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Strong Data Processing Inequality [AG76]: For any channel $V \in \mathbb{R}_{\text{sto}}^{q \times s}$,

$$\forall P_X, Q_X \in \mathcal{P}_q, \;\; \eta \, D(P_X || Q_X) \geq D(P_X V || Q_X V)$$

where $\eta \in [0, 1]$ is a channel dependent contraction coefficient.

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Relation to Erasure Channels [PW16]:
A q-ary erasure channel $E_{1-\eta} \in \mathbb{R}_{\mathsf{sto}}^{q \times (q+1)}$ erases its input with probability $1-\eta$, and keeps it the same with probability η.

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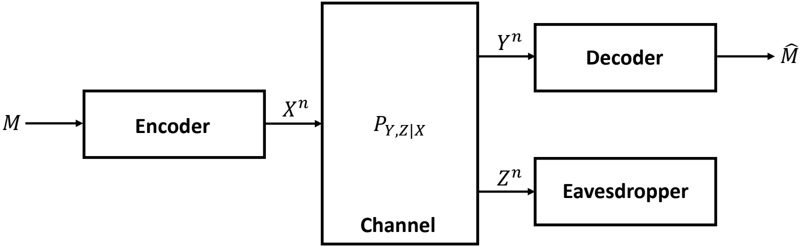
Prop: $E_{1-\eta} \succeq_{\text{ln}} V \ \ \Leftrightarrow \ \ \forall P_X, Q_X \in \mathcal{P}_q, \, \eta D(P_X || Q_X) \geq D(P_X V || Q_X V)$.

SDPI $= \succeq_{\text{ln}}$ domination by erasure channel

Motivation: Wyner's Wiretap Channel



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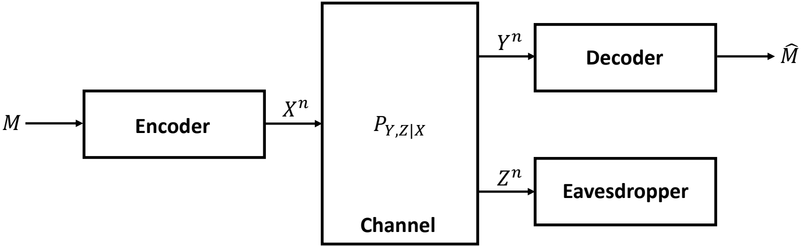

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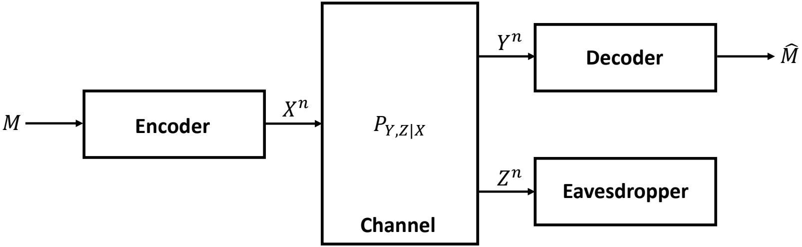

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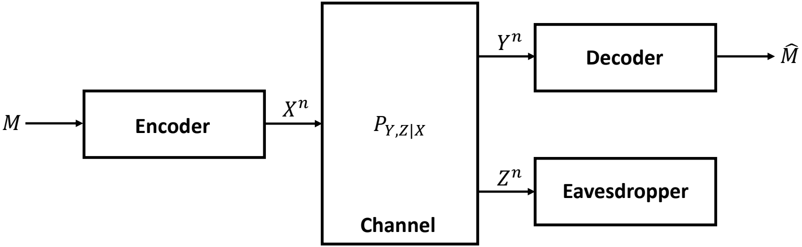

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- **Prop** [CK11]: $C_S = 0$ if and only if $W_\delta \succeq_{\text{ln}} V$.
- Finding the maximally noisy $W_\delta \succeq_{\text{ln}} V$ establishes the minimal noise on $P_{Z|X}$ so that secret communication is feasible.

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  A channel $V \in \mathbb{R}_{\text{sto}}^{q \times s}$ is a degraded version of $W \in \mathbb{R}_{\text{sto}}^{q \times r}$, denoted $W \succeq_{\text{deg}} V$, if $V = WA$ for some channel $A \in \mathbb{R}_{\text{sto}}^{r \times s}$.

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Theorem (Degradation by Symmetric Channels)

Given a channel $V \in \mathbb{R}_{\text{sto}}^{q \times q}$ with $q \geq 2$ and minimum probability $\nu = \min\left\{[V]_{i,j} : 1 \leq i,j \leq q\right\}$, we have:

$$0 \leq \delta \leq \frac{\nu}{1 - (q-1)\nu + \frac{\nu}{q-1}} \Rightarrow W_\delta \succeq_{\text{deg}} V .$$

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Tightness for Degradation: The condition is tight when no further information about V is known. For example, suppose:

$$V = \begin{bmatrix} \nu & 1-(q-1)\nu & \nu & \cdots & \nu \\ 1-(q-1)\nu & \nu & \nu & \cdots & \nu \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-(q-1)\nu & \nu & \nu & \cdots & \nu \end{bmatrix} \in \mathbb{R}_{\text{sto}}^{q \times q}.$$

Then, $W_\delta \succeq_{\text{deg}} V$ if and only if $0 \leq \delta \leq \nu / \left(1 - (q-1)\nu + \frac{\nu}{q-1}\right)$.

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- The channel transition probability matrix is a doubly stochastic $\mathcal{X}$-circulant matrix $\text{circ}_{\mathcal{X}}(P_Z) \in \mathbb{R}_{\text{sto}}^{q \times q}$ defined entry-wise as:

$$\forall x, y \in \mathcal{X}, \;\; [\text{circ}_{\mathcal{X}}(P_Z)]_{x,y} \triangleq P_Z(-x \oplus y) = P_{Y|X}(y|x).$$

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- Symmetric channel: $P_Z = \left(1 - \delta, \frac{\delta}{q-1}, \dots, \frac{\delta}{q-1}\right)$ for $\delta \in [0, 1]$

$$\text{circ}_{\mathcal{X}}(P_Z) = W_\delta$$

Less Noisy Domination and Degradation Regions

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Domination Structure of Additive Noise Channels

Theorem (Less Noisy Domination and Degradation Regions)

Given $W_\delta \in \mathbb{R}_{\text{sto}}^{q \times q}$ with $\delta \in \left[0, \frac{q-1}{q}\right]$ and $q \geq 2$, we have:

$$\begin{aligned} \mathcal{D}_{W_\delta}^{\text{add}} &= \text{conv}\left(\text{rows of } W_\delta\right) \\ &\subseteq \text{conv}\left(\text{rows of } W_\delta \text{ and } W_\gamma\right) \\ &\subseteq \mathcal{L}_{W_\delta}^{\text{add}} \subseteq \left\{P_Z \in \mathcal{P}_q : \left\|P_Z - \mathbf{u}\right\|_{\ell^2} \leq \left\|w_\delta - \mathbf{u}\right\|_{\ell^2}\right\} \end{aligned}$$

where w_δ is the first row of W_δ, $\gamma = (1-\delta)/\left(1 - \delta + \frac{\delta}{(q-1)^2}\right)$, and $\mathbf{u} \in \mathcal{P}_q$ is the uniform pmf.

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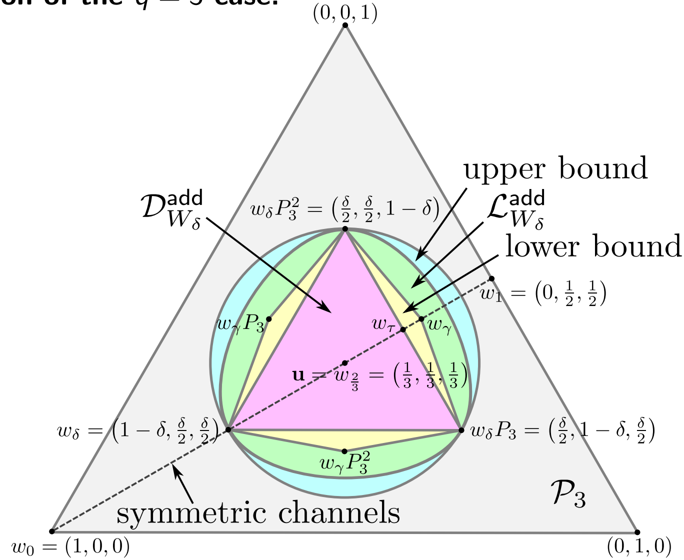
Furthermore, $\mathcal{L}_{W_\delta}^{\text{add}}$ is a closed and convex set that is symmetric with respect to permutations representing the group $(\mathcal{X}, \oplus)$.

Domination Structure of Additive Noise Channels

Illustration of the $q = 3$ case:



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2. Conditions for Domination by a Symmetric Channel
3. Less Noisy Domination and Log-Sobolev Inequalities
   - Log-Sobolev Inequalities
   - Comparison of Dirichlet Forms

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- The log-Sobolev inequality with constant $\alpha \in \mathbb{R}^+$ states that for every $f \in \mathbb{R}^q$ such that $f^T f = q$:

$$D\left(f^2 \mathbf{u} \,||\, \mathbf{u}\right) = \frac{1}{q} \sum_{i=1}^{q} f_i^2 \log\left(f_i^2\right) \leq \frac{1}{\alpha} \mathcal{E}_V(f, f)$$

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where the largest possible constant α satisfying this inequality is known as the log-Sobolev constant.

Comparison of Dirichlet Forms

- Log-Sobolev constant of the standard Dirichlet form:

$$\mathcal{E}_{\mathsf{std}}\left(f,f\right) \triangleq \mathbb{VAR}_{\mathbf{u}}(f) = \sum_{i=1}^{q} \frac{1}{q} f_i^2 - \left(\sum_{i=1}^{q} \frac{1}{q} f_i \right)^2$$

is known [DSC96]. For every $f \in \mathbb{R}^q$ with $f^T f = q$:

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Theorem (Domination of Dirichlet Forms)

For any channels $W_\delta \in \mathbb{R}_{\text{sto}}^{q\times q}$ with $\delta \in \left[0, \frac{q-1}{q}\right]$ and $V \in \mathbb{R}_{\text{sto}}^{q\times q}$, that have uniform stationary distribution, if $W_\delta \succeq_{\text{ln}} V$, then $\mathcal{E}_V \geq \frac{q\delta}{q-1} \mathcal{E}_{\text{std}}$ pointwise.

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- This establishes a log-Sobolev inequality for V:

$$D(f^2\mathbf{u}||\mathbf{u}) \leq \frac{(q-1)\log(q-1)}{\delta\,(q-2)} \mathcal{E}_V(f,f)$$

for every $f \in \mathbb{R}^q$ satisfying $f^T f = q$.

Thank You!