Permutation Channels

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Massachusetts Institute of Technology

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Outline

Three Motivations

- **Coding theory:** [DG01], [Mit06], [Met09], [KV15], [KT18], . . .

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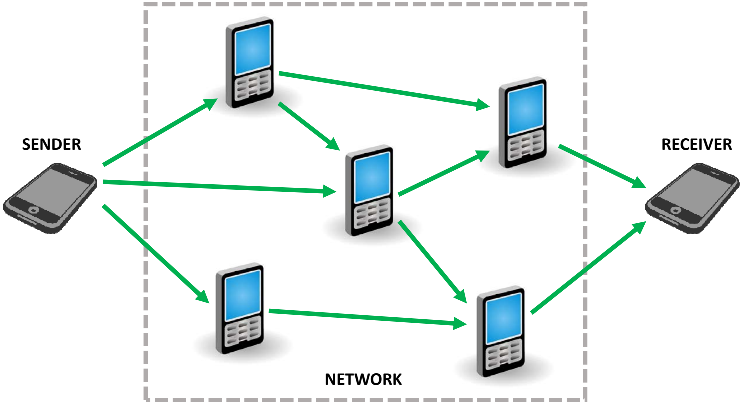

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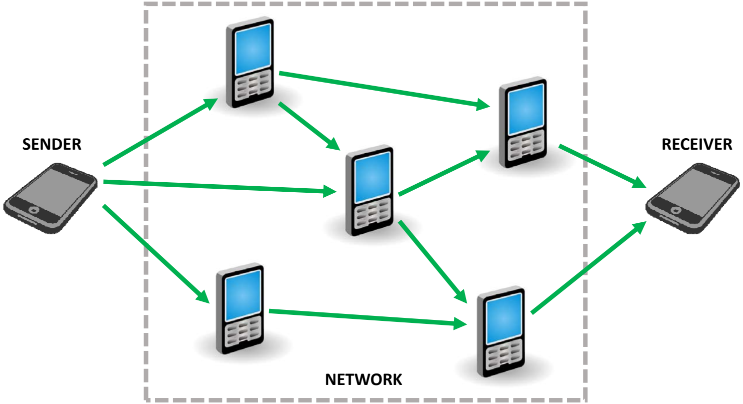

Model communication network as a channel

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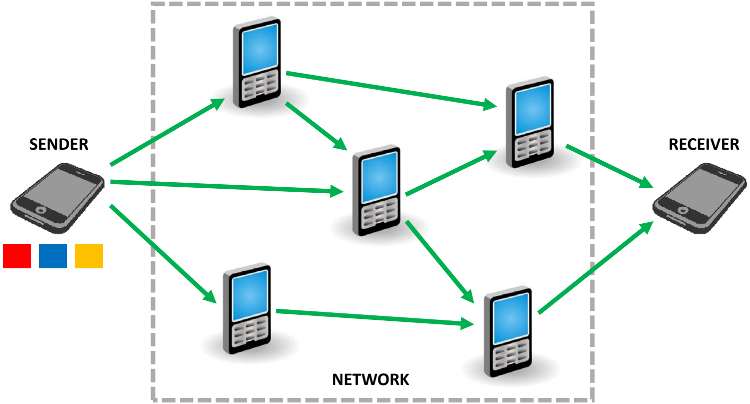

Model communication network as a channel:

- Alphabet symbols = all possible b-bit packets $\Rightarrow$ 2^b input symbols

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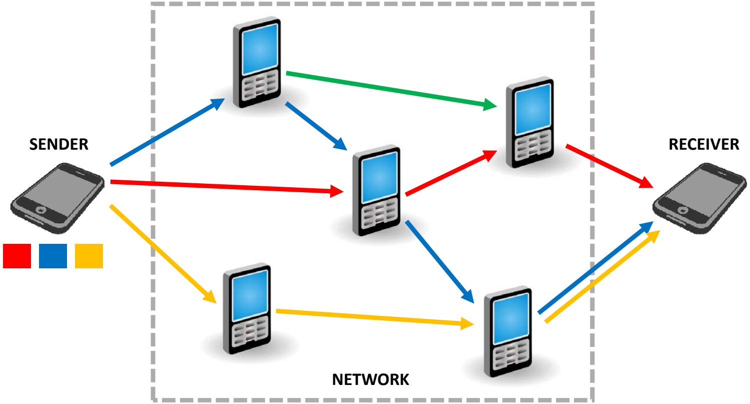

Model communication network as a channel:

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- Multipath routed network or evolving network topology

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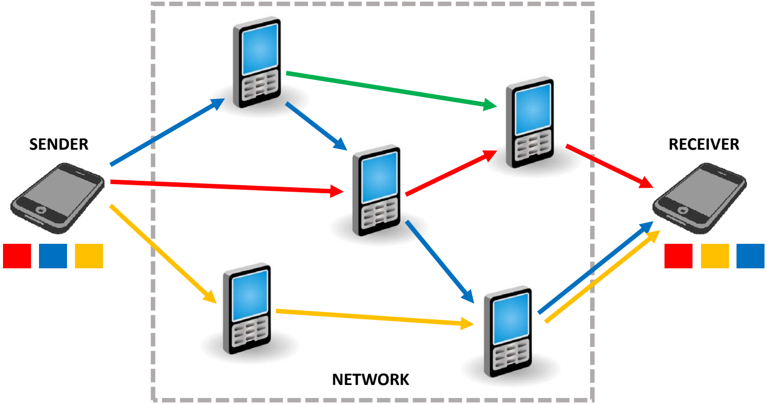

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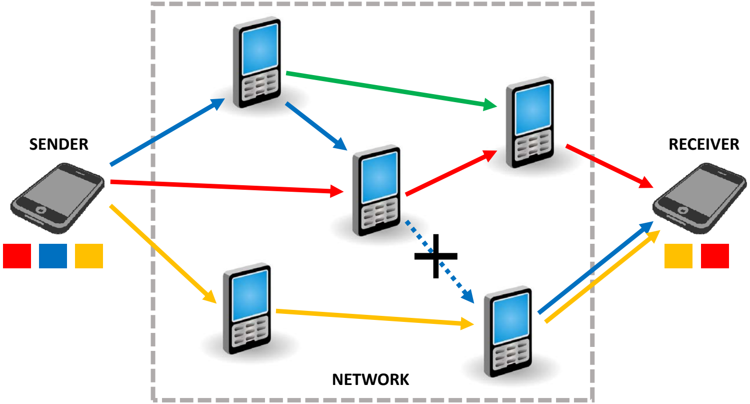

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- Packets are impaired (e.g., deletions, substitutions, etc.)

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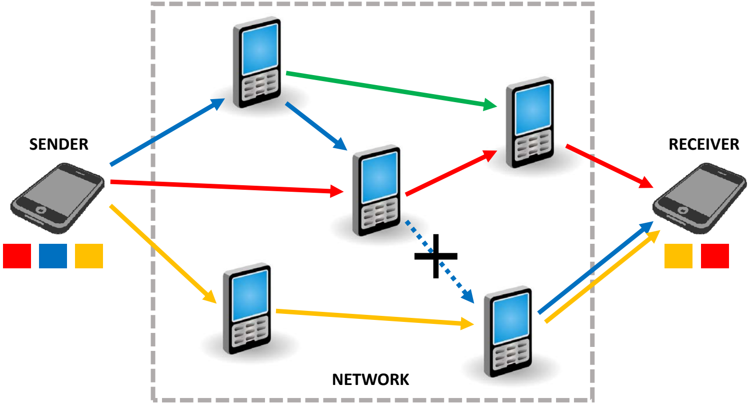

Model communication network as a channel:

- Alphabet symbols = all possible b-bit packets
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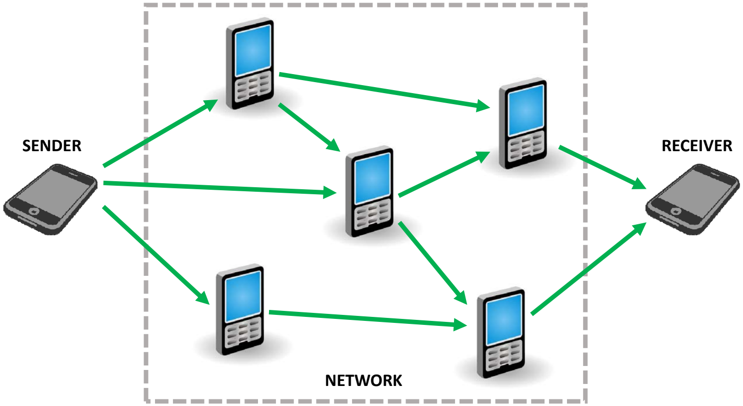
- Packets are impaired ⇒ model using channel probabilities

Example: Coding for Random Deletion Network

Consider a communication network where packets can be dropped:



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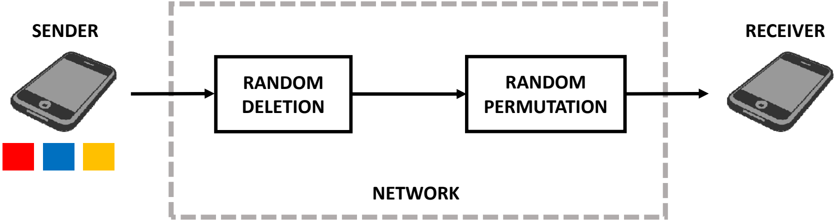

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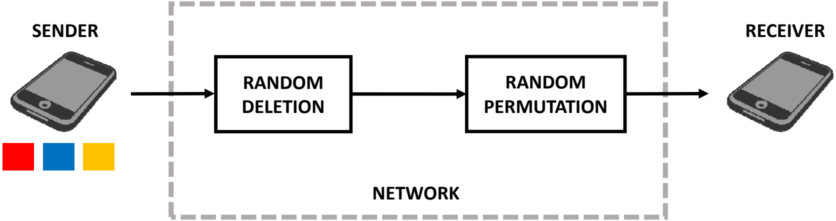

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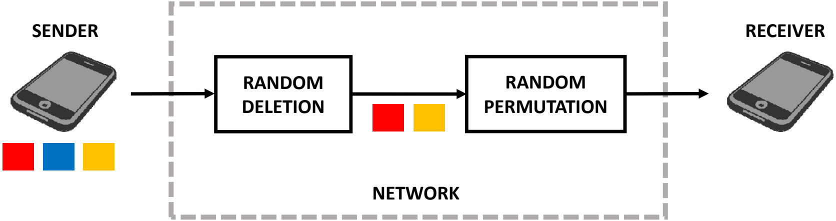

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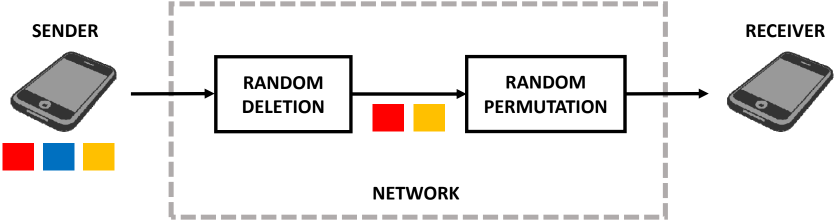

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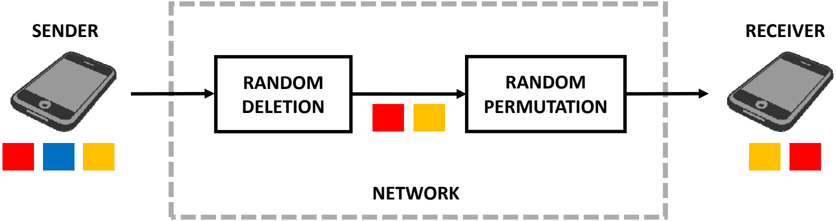

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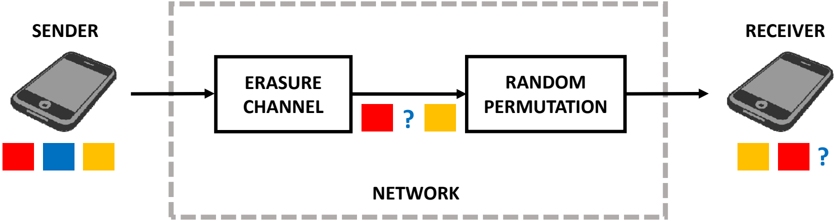

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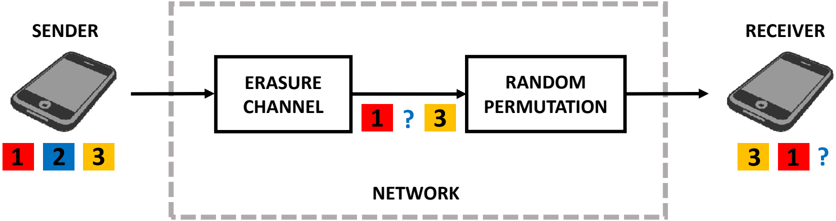

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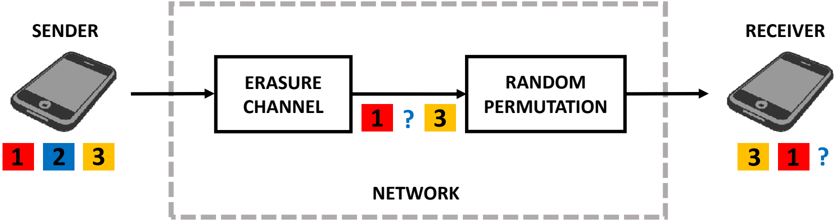

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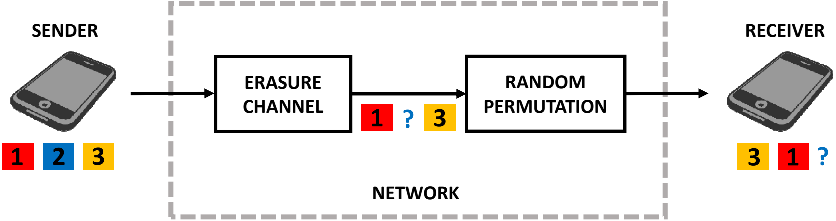

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- More refined coding techniques *simulate* sequence numbers [Mit06], [Met09]

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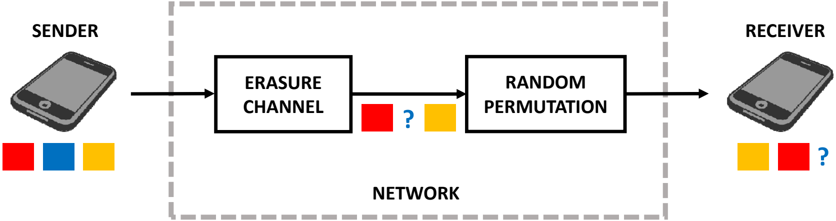

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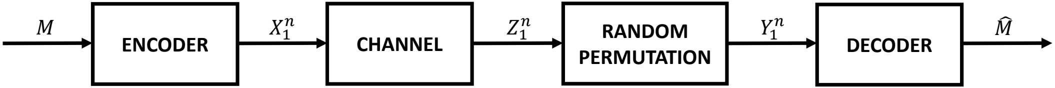
How do you code in such channels without increasing alphabet size?

Permutation Channel Model



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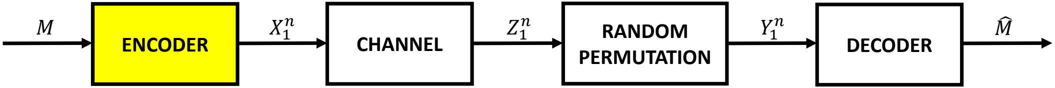

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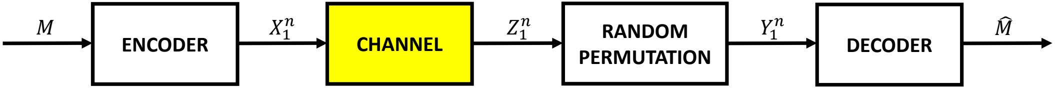

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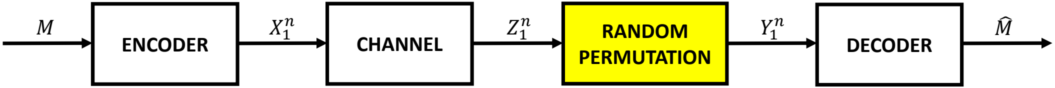

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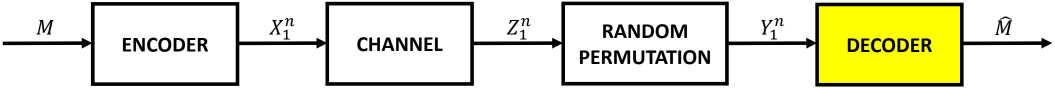

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- Randomized decoder $g_n : \mathcal{Y}^n \to \mathcal{M} \cup \{\text{error}\}$ produces estimate $\hat{M} = g_n(Y_1^n)$ at receiver

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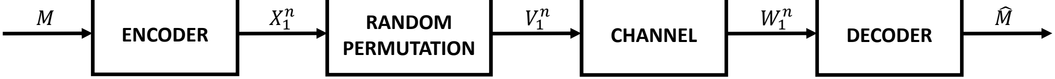
What if we analyze the "swapped" model?



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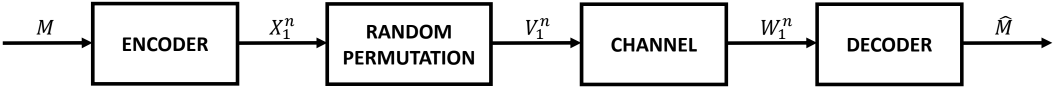

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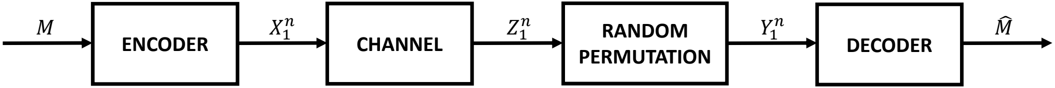

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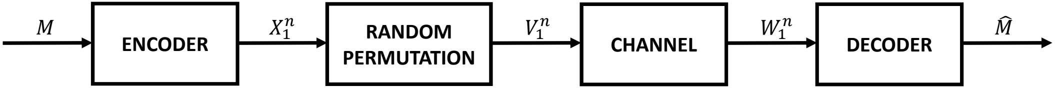

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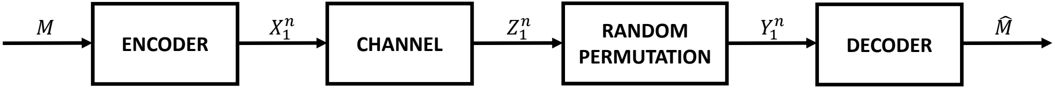

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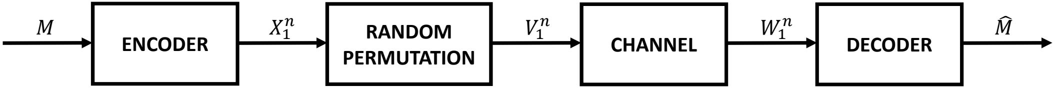

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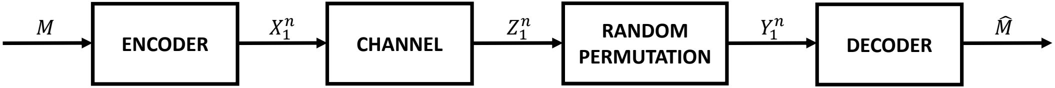

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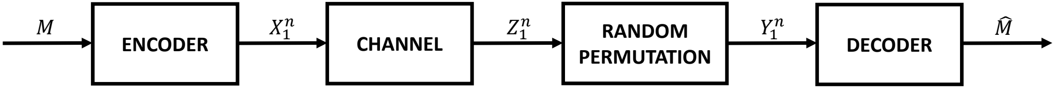

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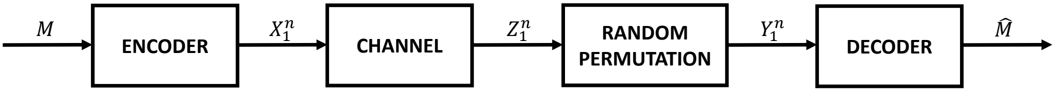

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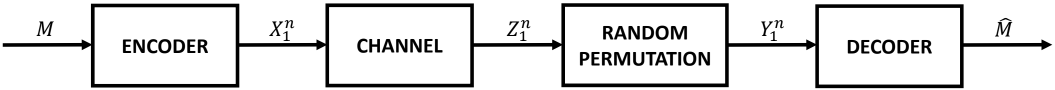

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What are the fundamental information theoretic limits of this model?

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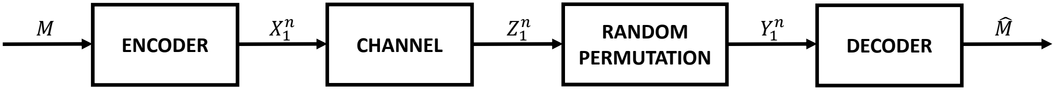

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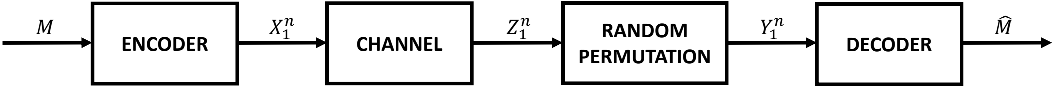

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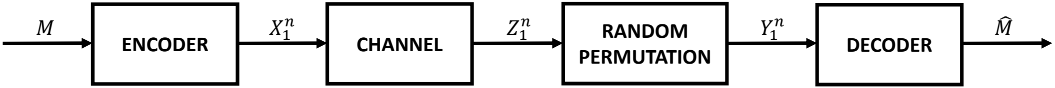

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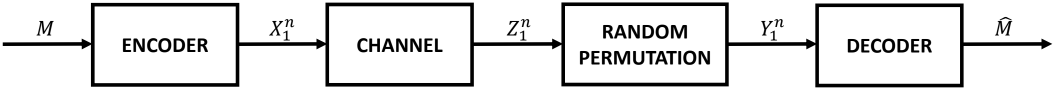

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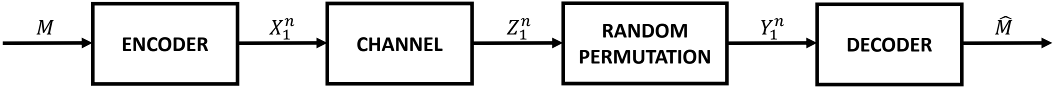
- $|\mathcal{M}| = n^R$ because number of empirical distributions of Y_1^n is $poly(n)$

Information Capacity of the Permutation Channel



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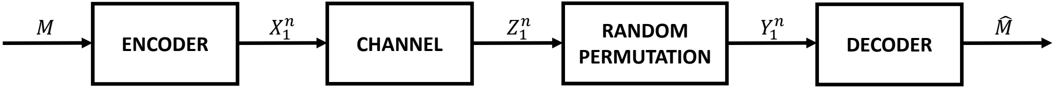

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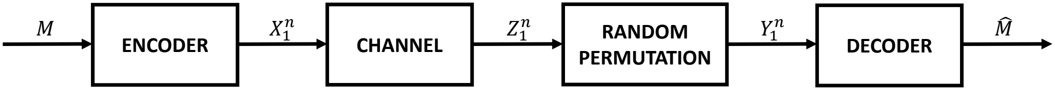

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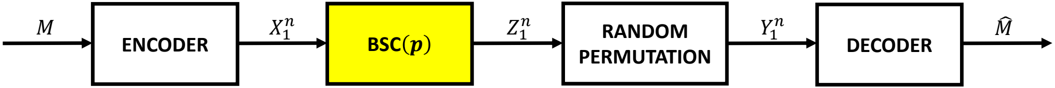
Main Question

What is the permutation channel capacity of a general $P_{Z|X}$?

Example: Binary Symmetric Channel



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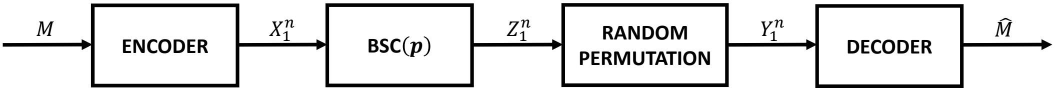

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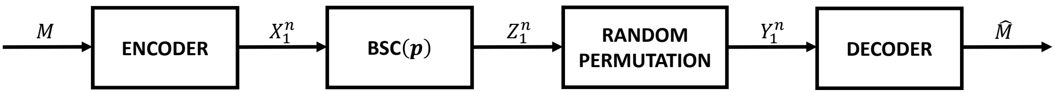

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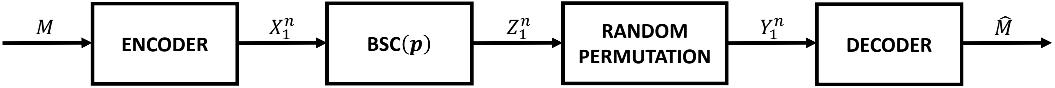

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Outline

1. Introduction
2. Achievability and Converse for the BSC
   - Encoder and Decoder
   - Testing between Converging Hypotheses
   - Second Moment Method for TV Distance
   - Fano's Inequality and CLT Approximation
3. General Achievability Bound
4. General Converse Bounds
5. Conclusion

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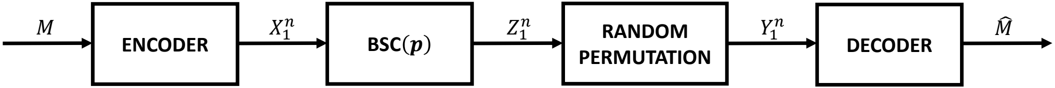

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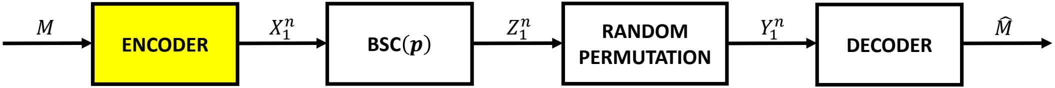

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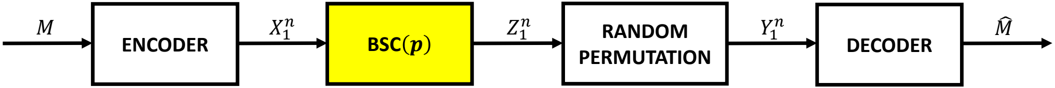

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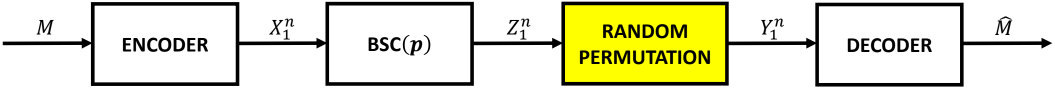

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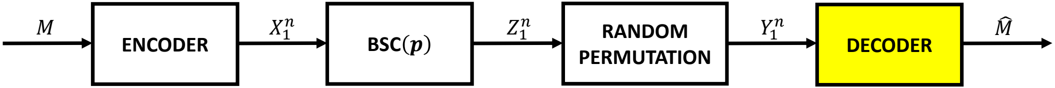

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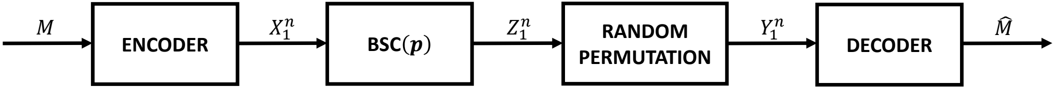

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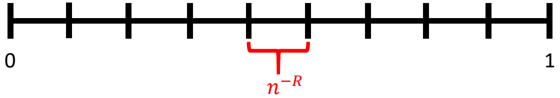

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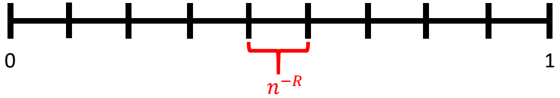

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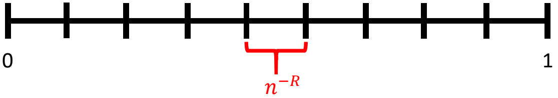

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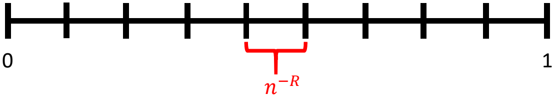

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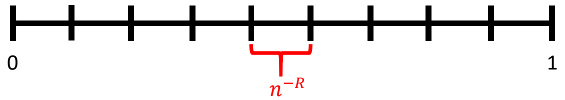

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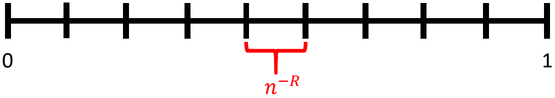

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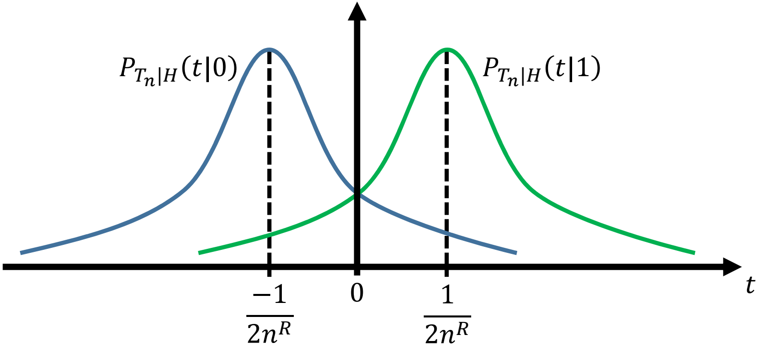
Figure:



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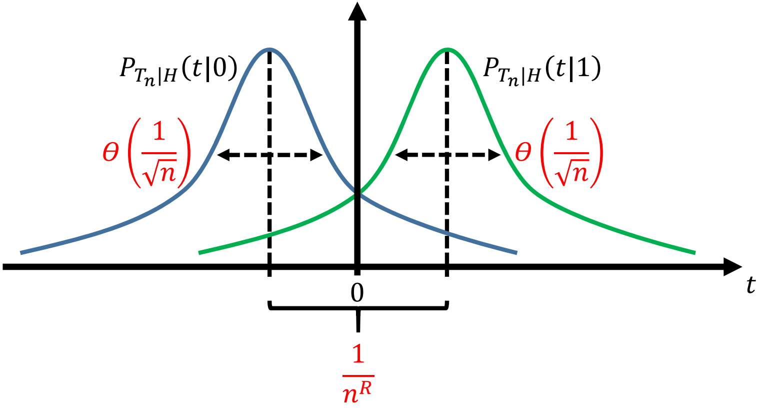

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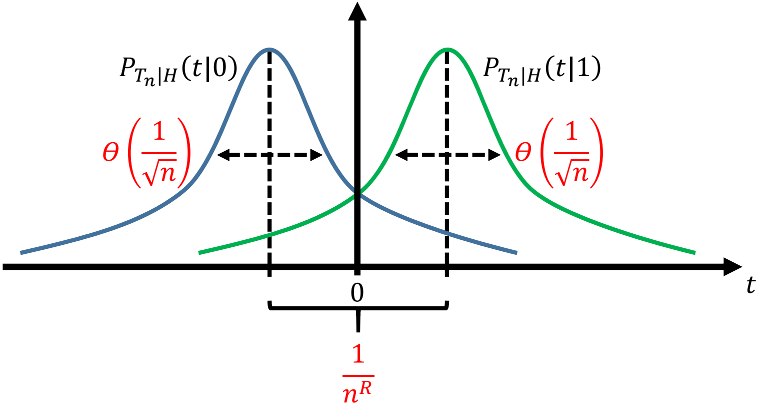

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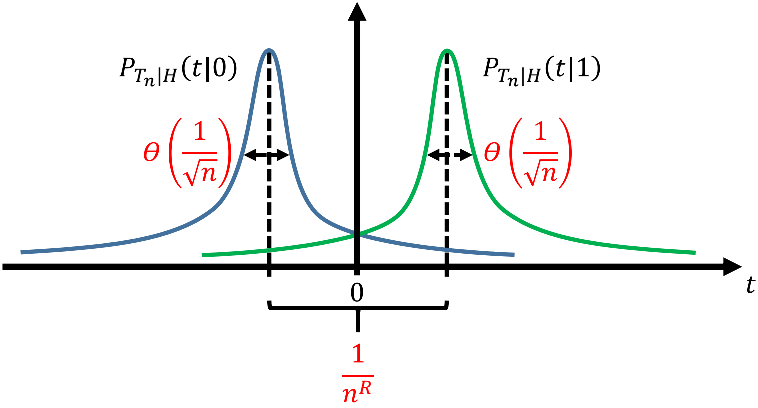

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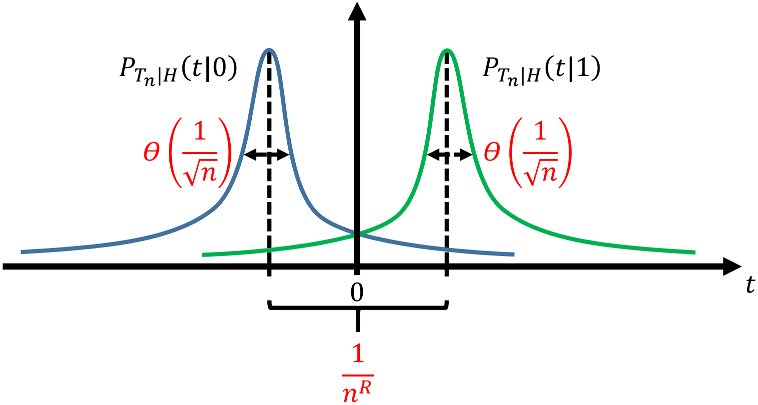

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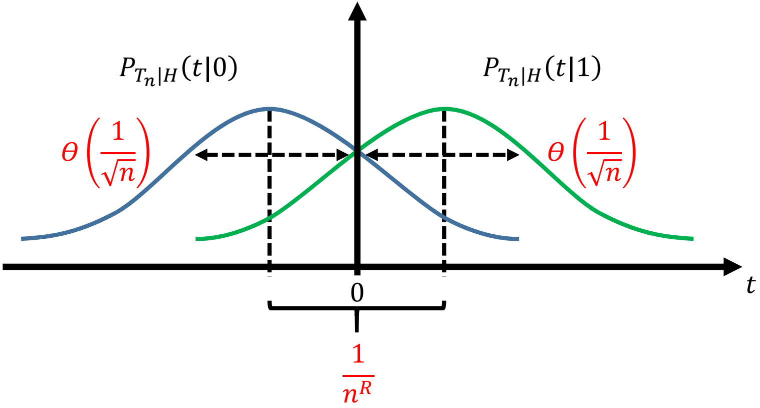

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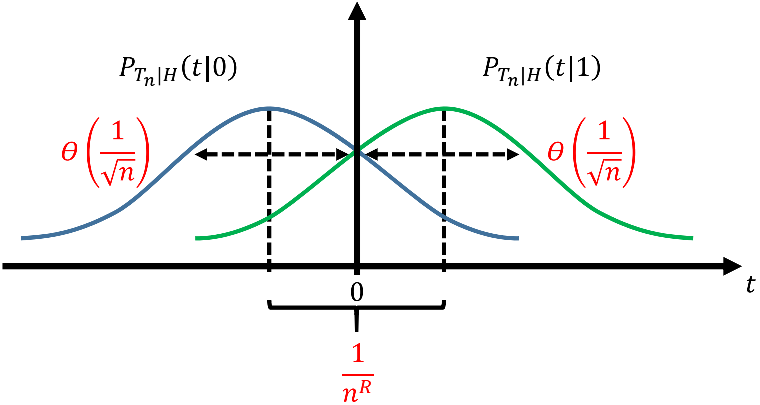

Second Moment Method for TV Distance

Lemma (2nd Moment Method [EKPS00])

$$\left\| P_{T_n|H=1} - P_{T_n|H=0} \right\|_{\mathsf{TV}} \geq \frac{\left(\mathbb{E}[T_n|H=1] - \mathbb{E}[T_n|H=0]\right)^2}{4\,\mathbb{VAR}(T_n)}$$

where $\|P - Q\|_{\mathsf{TV}} = \frac{1}{2}\|P - Q\|_1$ denotes the *total variation (TV) distance* between the distributions P and Q.

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Proof: Let $T_n^+ \sim P_{T_n|H=1}$ and $T_n^- \sim P_{T_n|H=0}$

$$\left(\mathbb{E}\left[T_n^+\right] - \mathbb{E}\left[T_n^-\right]\right)^2 = \left(\sum_t t\left(P_{T_n|H}(t|1) - P_{T_n|H}(t|0)\right)\right)^2$$

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$$\left(\mathbb{E}\left[T_n^+\right] - \mathbb{E}\left[T_n^-\right]\right)^2 = \left(\sum_t t\sqrt{P_{T_n}(t)}\,\frac{\left(P_{T_n|H}(t|1) - P_{T_n|H}(t|0)\right)}{\sqrt{P_{T_n}(t)}}\right)^2$$

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where $\|P - Q\|_{\mathsf{TV}} = \frac{1}{2}\|P - Q\|_1$ denotes the *total variation (TV) distance* between the distributions P and Q.

Proof: Cauchy-Schwarz inequality

$$\left(\mathbb{E}\left[T_n^+\right] - \mathbb{E}\left[T_n^-\right]\right)^2 = \left(\sum_t t\sqrt{P_{T_n}(t)}\,\frac{\left(P_{T_n|H}(t|1) - P_{T_n|H}(t|0)\right)}{\sqrt{P_{T_n}(t)}}\right)^2$$

$$\leq \left(\sum_t t^2 P_{T_n}(t)\right)\left(\sum_t \frac{\left(P_{T_n|H}(t|1) - P_{T_n|H}(t|0)\right)^2}{P_{T_n}(t)}\right)$$

Second Moment Method for TV Distance

Lemma (2nd Moment Method [EKPS00])

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where $\|P - Q\|_{\mathsf{TV}} = \frac{1}{2}\|P - Q\|_1$ denotes the *total variation (TV) distance* between the distributions P and Q.

Proof: Recall that T_n is *zero-mean*

$$\left(\mathbb{E}\left[T_n^+\right] - \mathbb{E}\left[T_n^-\right]\right)^2 = \left(\sum_t t\sqrt{P_{T_n}(t)}\,\frac{\left(P_{T_n|H}(t|1) - P_{T_n|H}(t|0)\right)}{\sqrt{P_{T_n}(t)}}\right)^2$$

$$\leq \mathbb{VAR}(T_n)\left(\sum_t \frac{\left(P_{T_n|H}(t|1) - P_{T_n|H}(t|0)\right)^2}{P_{T_n}(t)}\right)$$

Second Moment Method for TV Distance

Lemma (2nd Moment Method [EKPS00])

$$\left\|P_{T_n|H=1} - P_{T_n|H=0}\right\|_{\mathsf{TV}} \geq \frac{(\mathbb{E}[T_n|H=1] - \mathbb{E}[T_n|H=0])^2}{4\,\mathbb{VAR}(T_n)}$$

where $\|P - Q\|_{\mathsf{TV}} = \frac{1}{2}\|P - Q\|_1$ denotes the *total variation (TV) distance* between the distributions P and Q.

Proof: Hammersley-Chapman-Robbins bound

$$\left(\mathbb{E}[T_n^+] - \mathbb{E}[T_n^-]\right)^2 = \left(\sum_t t\sqrt{P_{T_n}(t)}\,\frac{\left(P_{T_n|H}(t|1) - P_{T_n|H}(t|0)\right)}{\sqrt{P_{T_n}(t)}}\right)^2$$

$$\leq 4\,\mathbb{VAR}(T_n)\underbrace{\left(\frac{1}{4}\sum_t \frac{\left(P_{T_n|H}(t|1) - P_{T_n|H}(t|0)\right)^2}{P_{T_n}(t)}\right)}_{\text{Vincze-Le Cam distance}}$$

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Proof:

$$\left(\mathbb{E}\left[T_n^+\right] - \mathbb{E}\left[T_n^-\right]\right)^2 = \left(\sum_t t\sqrt{P_{T_n}(t)}\,\frac{\left(P_{T_n|H}(t|1) - P_{T_n|H}(t|0)\right)}{\sqrt{P_{T_n}(t)}}\right)^2$$

$$\leq 4\,\mathbb{VAR}(T_n)\left(\frac{1}{4}\sum_t \frac{\left(P_{T_n|H}(t|1) - P_{T_n|H}(t|0)\right)^2}{P_{T_n}(t)}\right)$$

$$\leq 4\,\mathbb{VAR}(T_n)\left\|P_{T_n|H=1} - P_{T_n|H=0}\right\|_{\mathsf{TV}}$$

BSC Achievability Proof

Proposition (BSC Achievability)

For any $0 < R < 1/2$, consider the binary hypothesis testing problem with $H \sim \text{Ber}\left(\frac{1}{2}\right)$, and $X_1^n \overset{\text{i.i.d.}}{\sim} \text{Ber}\left(q + \frac{h}{n^R}\right)$ given $H = h \in \{0, 1\}$.

Proof: Start with Le Cam's relation

$$P_{\text{ML}}^n = \frac{1}{2}\left(1 - \left\|P_{T_n|H=1} - P_{T_n|H=0}\right\|_{\text{TV}}\right)$$

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Proof: Apply second moment method lemma

$$P_{\text{ML}}^n = \frac{1}{2}\left(1 - \left\|P_{T_n|H=1} - P_{T_n|H=0}\right\|_{\text{TV}}\right)$$
$$\leq \frac{1}{2}\left(1 - \frac{(\mathbb{E}[T_n|H=1] - \mathbb{E}[T_n|H=0])^2}{4\,\mathbb{VAR}(T_n)}\right)$$

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Proof: After explicit computation and simplification...

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Then, $\lim_{n\to\infty} P_{\text{ML}}^n = 0$.

Proof: For any $0 < R < \frac{1}{2}$,

$$\begin{aligned} P_{\text{ML}}^n &= \frac{1}{2}\left(1 - \left\|P_{T_n|H=1} - P_{T_n|H=0}\right\|_{\text{TV}}\right) \\ &\le \frac{1}{2}\left(1 - \frac{\left(\mathbb{E}[T_n|H=1] - \mathbb{E}[T_n|H=0]\right)^2}{4\,\mathbb{VAR}(T_n)}\right) \\ &\le \frac{3}{2n^{1-2R}} \to 0 \text{ as } n \to \infty \end{aligned}$$

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Then, $\lim_{n\to\infty} P_{\text{ML}}^n = 0$. This implies that:

$$C_{\text{perm}}(\text{BSC}(p)) \geq \frac{1}{2} .$$

Proof: For any $0 < R < \frac{1}{2}$,

$$\begin{aligned}
P_{\text{ML}}^n &= \frac{1}{2}\left(1 - \left\|P_{T_n|H=1} - P_{T_n|H=0}\right\|_{\text{TV}}\right) \\
&\leq \frac{1}{2}\left(1 - \frac{\left(\mathbb{E}[T_n|H=1] - \mathbb{E}[T_n|H=0]\right)^2}{4\,\mathbb{VAR}(T_n)}\right) \\
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\end{aligned}$$

Outline

Recall: Basic Definitions of Information Measures

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$$= H(X) - H(X|Y)$$

Recall: Two Information Inequalities

Consider discrete random variables X, Y, Z that form a Markov chain $X \to Y \to Z$.

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with equality if and only if Z is a *sufficient statistic* of Y for X, i.e., $X \to Z \to Y$ also forms a Markov chain.

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with equality if and only if Z is a *sufficient statistic* of Y for X, i.e., $X \to Z \to Y$ also forms a Markov chain.

Lemma (Fano's Inequality [CT06])

If X takes values in the finite alphabet $\mathcal{X}$, then

$$H(X|Z) \leq 1 + \mathbb{P}(X \neq Z) \log(|\mathcal{X}|)$$

where we perceive Z as an estimator for X based on Y.

BSC Converse Proof: Fano's Inequality Argument

- Consider the Markov chain $M \to X_1^n \to Z_1^n \to Y_1^n \to S_n \triangleq \sum_{i=1}^n Y_i \to \hat{M}$, and a sequence of encoder-decoder pairs $\{(f_n, g_n)\}_{n\in\mathbb{N}}$ such that $|\mathcal{M}| = n^R$ and $\lim_{n\to\infty} P_{\text{error}}^n = 0$

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- Standard argument [CT06]: M is uniform

$$R \log(n) = H(M)$$

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- Standard argument [CT06]: Fano's inequality, data processing inequality

$$
\begin{aligned}
R \log(n) &= H(M|\hat{M}) + I(M; \hat{M}) \\
&\leq 1 + P_{\text{error}}^n R \log(n) + I(M; Y_1^n)
\end{aligned}
$$

BSC Converse Proof: Fano's Inequality Argument

- Consider the Markov chain $M \to X_1^n \to Z_1^n \to Y_1^n \to S_n \triangleq \sum_{i=1}^n Y_i \to \hat{M}$, and a sequence of encoder-decoder pairs $\{(f_n, g_n)\}_{n\in\mathbb{N}}$ such that $|\mathcal{M}| = n^R$ and $\lim_{n\to\infty} P_{\text{error}}^n = 0$
- Standard argument [CT06]: sufficiency

$$\begin{aligned} R\log(n) &= H(M|\hat{M}) + I(M;\hat{M}) \\ &\leq 1 + P_{\text{error}}^n R\log(n) + I(M;Y_1^n) \\ &= 1 + P_{\text{error}}^n R\log(n) + I(M;S_n) \end{aligned}$$

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- Standard argument [CT06]: data processing inequality

$$\begin{aligned} R\log(n) &= H(M|\hat{M}) + I(M;\hat{M}) \\ &\leq 1 + P_{\text{error}}^n R\log(n) + I(M;Y_1^n) \\ &= 1 + P_{\text{error}}^n R\log(n) + I(M;S_n) \\ &\leq 1 + P_{\text{error}}^n R\log(n) + I(X_1^n;S_n) \end{aligned}$$

BSC Converse Proof: Fano's Inequality Argument

- Consider the Markov chain $M \rightarrow X_1^n \rightarrow Z_1^n \rightarrow Y_1^n \rightarrow S_n \triangleq \sum_{i=1}^n Y_i \rightarrow \hat{M}$, and a sequence of encoder-decoder pairs $\{(f_n, g_n)\}_{n \in \mathbb{N}}$ such that $|\mathcal{M}| = n^R$ and $\lim_{n \to \infty} P_{\text{error}}^n = 0$
- Standard argument [CT06]:

$$\begin{aligned} R \log(n) &= H(M|\hat{M}) + I(M; \hat{M}) \\ &\leq 1 + P_{\text{error}}^n R \log(n) + I(M; Y_1^n) \\ &= 1 + P_{\text{error}}^n R \log(n) + I(M; S_n) \\ &\leq 1 + P_{\text{error}}^n R \log(n) + I(X_1^n; S_n) \end{aligned}$$

- Divide by $\log(n)$:

$$R \leq \frac{1}{\log(n)} + P_{\text{error}}^n R + \frac{I(X_1^n; S_n)}{\log(n)}$$

BSC Converse Proof: Fano's Inequality Argument

- Consider the Markov chain $M \to X_1^n \to Z_1^n \to Y_1^n \to S_n \triangleq \sum_{i=1}^n Y_i \to \hat{M}$, and a sequence of encoder-decoder pairs $\{(f_n, g_n)\}_{n\in\mathbb{N}}$ such that $|\mathcal{M}| = n^R$ and $\lim_{n\to\infty} P_{\text{error}}^n = 0$

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- Divide by $\log(n)$ and let $n \to \infty$:

$$R \leq \lim_{n\to\infty} \frac{I(X_1^n;S_n)}{\log(n)}$$

Upper bound on $I(X_1^n; S_n)$:

$$I(X_1^n; S_n) = H(S_n) - H(S_n | X_1^n)$$

BSC Converse Proof: CLT Approximation

Since $S_n \in \{0, \ldots, n\}$,

$$I(X_1^n; S_n) = H(S_n) - H(S_n | X_1^n)$$
$$\leq \log(n+1) - \sum_{x_1^n \in \{0,1\}^n} P_{X_1^n}(x_1^n) \, H(S_n | X_1^n = x_1^n)$$

BSC Converse Proof: CLT Approximation

Given $X_1^n = x_1^n$ with $\sum_{i=1}^n x_i = k$, $S_n = \text{bin}(k, 1-p) + \text{bin}(n-k, p)$:

$$I(X_1^n; S_n) = H(S_n) - H(S_n|X_1^n)$$
$$\leq \log(n+1) - \sum_{x_1^n \in \{0,1\}^n} P_{X_1^n}(x_1^n)\, H(\text{bin}(k, 1-p) + \text{bin}(n-k, p))$$

BSC Converse Proof: CLT Approximation

Using [CT06, Problem 2.14], i.e., $\max\{H(X), H(Y)\} \leq H(X+Y)$ for $X \perp\!\!\!\perp Y$,

$$\begin{aligned} I(X_1^n; S_n) &= H(S_n) - H(S_n|X_1^n) \\ &\leq \log(n+1) - \sum_{x_1^n \in \{0,1\}^n} P_{X_1^n}(x_1^n)\, H(\text{bin}(k, 1-p) + \text{bin}(n-k, p)) \\ &\leq \log(n+1) - \sum_{x_1^n \in \{0,1\}^n} P_{X_1^n}(x_1^n)\, H\left(\text{bin}\left(\frac{n}{2}, p\right)\right) \end{aligned}$$

BSC Converse Proof: CLT Approximation

Approximate binomial entropy using CLT [ALY10]:

$$
\begin{aligned}
I(X_1^n; S_n) &= H(S_n) - H(S_n | X_1^n) \\
&\leq \log(n+1) - \sum_{x_1^n \in \{0,1\}^n} P_{X_1^n}(x_1^n)\, H(\mathrm{bin}(k, 1-p) + \mathrm{bin}(n-k, p)) \\
&\leq \log(n+1) - \sum_{x_1^n \in \{0,1\}^n} P_{X_1^n}(x_1^n)\, H\left(\mathrm{bin}\left(\frac{n}{2}, p\right)\right) \\
&= \log(n+1) - \sum_{x_1^n \in \{0,1\}^n} P_{X_1^n}(x_1^n) \left(\frac{1}{2}\log(\pi e p(1-p)n) + O\left(\frac{1}{n}\right)\right)
\end{aligned}
$$

BSC Converse Proof: CLT Approximation

Upper bound on $I(X_1^n; S_n)$:

$$\begin{aligned} I(X_1^n; S_n) &= H(S_n) - H(S_n | X_1^n) \\ &\leq \log(n+1) - \sum_{x_1^n \in \{0,1\}^n} P_{X_1^n}(x_1^n)\, H(\text{bin}(k, 1-p) + \text{bin}(n-k, p)) \\ &\leq \log(n+1) - \sum_{x_1^n \in \{0,1\}^n} P_{X_1^n}(x_1^n)\, H\left(\text{bin}\left(\frac{n}{2}, p\right)\right) \\ &= \log(n+1) - \frac{1}{2}\log(\pi e p(1-p)n) + O\left(\frac{1}{n}\right) \end{aligned}$$

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Hence, we have $R \leq \lim_{n\to\infty} I(X_1^n; S_n)/\log(n) = \frac{1}{2}$.

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Upper bound on $I(X_1^n; S_n)$:

$$\begin{aligned} I(X_1^n; S_n) &= H(S_n) - H(S_n|X_1^n) \\ &\leq \log(n+1) - \sum_{x_1^n \in \{0,1\}^n} P_{X_1^n}(x_1^n)\, H(\text{bin}(k, 1-p) + \text{bin}(n-k, p)) \\ &\leq \log(n+1) - \sum_{x_1^n \in \{0,1\}^n} P_{X_1^n}(x_1^n)\, H\left(\text{bin}\left(\frac{n}{2}, p\right)\right) \\ &= \log(n+1) - \frac{1}{2}\log(\pi e p(1-p)n) + O\left(\frac{1}{n}\right) \end{aligned}$$

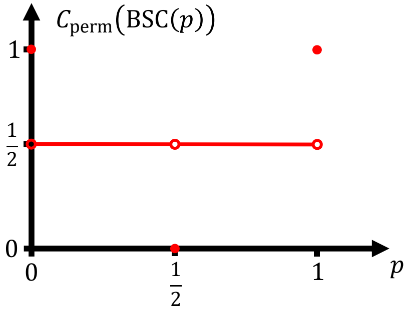
Hence, we have $R \leq \lim_{n\to\infty} I(X_1^n; S_n)/\log(n) = \frac{1}{2}$.

Proposition (BSC Converse)

$$C_{\text{perm}}(\text{BSC}(p)) \leq \frac{1}{2}$$

Proposition (Pemutation Channel Capacity of BSC)

$$C_{\text{perm}}(\mathsf{BSC}(p)) = \begin{cases} 1, & \text{for } p = 0, 1 \\ \frac{1}{2}, & \text{for } p \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \\ 0, & \text{for } p = \frac{1}{2} \end{cases}$$



Information Capacity of the BSC Permutation Channel

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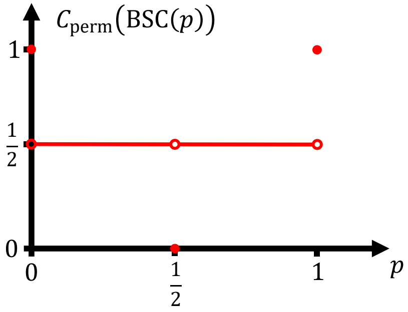

Remarks:

- $C_{\text{perm}}(\cdot)$ is discontinuous and non-convex

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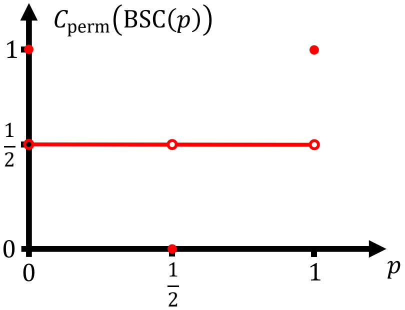

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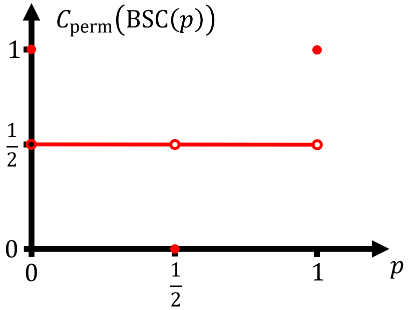
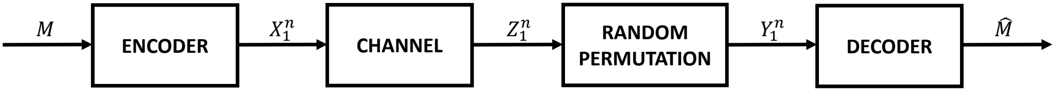


Remarks:

- $C_{\text{perm}}(\cdot)$ is discontinuous and non-convex
- $C_{\text{perm}}(\cdot)$ is generally agnostic to parameters of channel
- Computationally tractable coding scheme in achievability proof

Outline

Recall General Problem



- Average probability of error $P_{\text{error}}^n \triangleq \mathbb{P}(M \neq \hat{M})$
- "Rate" of coding scheme (f_n, g_n) is $R \triangleq \dfrac{\log(|\mathcal{M}|)}{\log(n)}$
- Rate $R \geq 0$ is achievable $\Leftrightarrow \exists \{(f_n, g_n)\}_{n \in \mathbb{N}}$ such that $\lim_{n\to\infty} P_{\text{error}}^n = 0$

Definition (Permutation Channel Capacity)

$$C_{\text{perm}}(P_{Z|X}) \triangleq \sup\{R \geq 0 : R \text{ is achievable}\}$$

Main Question

What is the permutation channel capacity of a general $P_{Z|X}$?

Achievability: Coding Scheme

- Let $r = \text{rank}(P_{Z|X})$ and $k = \lfloor \sqrt{n} \rfloor$

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- **Message set:**

$$\mathcal{M} \triangleq \left\{ p = (p(x) : x \in \mathcal{X}') \in (\mathbb{Z}_+)^{\mathcal{X}'} : \sum_{x \in \mathcal{X}'} p(x) = k \right\}$$

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- **Randomized Encoder:**

$$\forall p \in \mathcal{M}, \quad f_n(p) = X_1^n \overset{\text{i.i.d.}}{\sim} P_X \quad \text{where} \quad P_X(x) = \begin{cases} \frac{p(x)}{k}, & \text{for } x \in \mathcal{X}' \\ 0, & \text{for } x \in \mathcal{X} \backslash \mathcal{X}' \end{cases}$$

Achievability: Coding Scheme

- Let stochastic matrix $\tilde{P}_{Z|X} \in \mathbb{R}^{r \times |\mathcal{Y}|}$ have rows $\{P_{Z|X}(\cdot|x) : x \in \mathcal{X}'\}$
- Let $\tilde{P}_{Z|X}^{\dagger}$ denote its *Moore-Penrose pseudoinverse*

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- **(Sub-optimal) Thresholding Decoder:** For any $y_1^n \in \mathcal{Y}^n$,
  Step 1: Construct its type/empirical distribution/histogram

$$\forall y \in \mathcal{Y}, \quad \hat{P}_{y_1^n}(y) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{y_i = y\}$$

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  Step 3: Output decoded message

$$g_n(y_1^n) = \begin{cases} \hat{p}, & \text{if } \hat{p} \in \mathcal{M} \\ \text{error}, & \text{otherwise} \end{cases}$$

Achievability: Rank Bound

Theorem (Rank Bound)

For any channel $P_{Z|X}$:

$$C_{\text{perm}}(P_{Z|X}) \geq \frac{\text{rank}(P_{Z|X}) - 1}{2} .$$

Remarks about Coding Scheme:

- Showing $\lim_{n\to\infty} P^n_{\text{error}} = 0$ proves theorem.

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- *Computational complexity*: Decoder has $O(n)$ running time.
- *Probabilistic method*: Good deterministic codes exist.
- *Expurgation*: Achievability bound holds under maximal probability of error criterion.

Outline

Converse: Output Alphabet Bound

Theorem (Output Alphabet Bound)

For any entry-wise *strictly positive* channel $P_{Z|X} > 0$:

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- **Want:** Converse bound in terms of input alphabet size.

Converse: Effective Input Alphabet Bound

Theorem (Effective Input Alphabet Bound)

For any entry-wise *strictly positive* channel $P_{Z|X} > 0$:

$$C_{\text{perm}}(P_{Z|X}) \leq \frac{\text{ext}(P_{Z|X}) - 1}{2}$$

where $\text{ext}(P_{Z|X})$ denotes the number of *extreme points* of $\text{conv}\{P_{Z|X}(\cdot|x) : x \in \mathcal{X}\}$.

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Remarks:

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- **How do we prove above theorem?**

Brief Digression: Degradation

Definition (Degradation/Blackwell Order [Bla51], [She51], [Ste51], [Cov72], [Ber73])

Given channels $P_{Z_1|X}$ and $P_{Z_2|X}$ with common input alphabet $\mathcal{X}$, $P_{Z_2|X}$ is a degraded version of $P_{Z_1|X}$ if $P_{Z_2|X} = P_{Z_1|X} P_{Z_2|Z_1}$ for some channel $P_{Z_2|Z_1}$.

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Theorem (Blackwell-Sherman-Stein [Bla51], [She51], [Ste51])

The *observation model* $P_{Z_2|X}$ is a degraded version of $P_{Z_1|X}$ if and only if for every prior distribution P_X, and every loss function $L : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, the *Bayes risks* satisfy:

$$\min_{f(\cdot)} \mathbb{E}\left[L(X, f(Z_1))\right] \leq \min_{g(\cdot)} \mathbb{E}\left[L(X, g(Z_2))\right]$$

where the minima are over all randomized estimators of X.

Brief Digression: Symmetric Channels

Definition (q-ary Symmetric Channel)

A q-ary symmetric channel, denoted q-SC(δ), with total crossover probability $\delta \in [0, 1]$ and alphabet $\mathcal{X}$ where $|\mathcal{X}| = q$, is given by the doubly stochastic matrix:

$$W_\delta \triangleq \begin{bmatrix} 1-\delta & \frac{\delta}{q-1} & \cdots & \frac{\delta}{q-1} \\ \frac{\delta}{q-1} & 1-\delta & \cdots & \frac{\delta}{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\delta}{q-1} & \frac{\delta}{q-1} & \cdots & 1-\delta \end{bmatrix}.$$

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Proposition (Degradation by Symmetric Channels)

Given channel $P_{Z|X}$ with $\nu = \min_{x \in \mathcal{X}, y \in \mathcal{Y}} P_{Z|X}(y|x)$,

if $0 \leq \delta \leq \dfrac{\nu}{1 - \nu + \frac{\nu}{q-1}}$, then $P_{Z|X}$ is a degraded version of q-SC(δ).

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Remarks:

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  - Additive noise channel on Abelian group $\mathcal{X}$ [MP18]: $\delta \leq (q-1)\nu$.

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  - Additive noise channel on Abelian group $\mathcal{X}$ [MP18]: $\delta \leq (q-1)\nu$.
  - Alternative bounds for Markov chains [MOS13].

Brief Digression: Symmetric Channels

Proposition (Degradation by Symmetric Channels)

Given channel $P_{Z|X}$ with $\nu = \min_{x \in \mathcal{X},\, y \in \mathcal{Y}} P_{Z|X}(y|x)$,

if $0 \leq \delta \leq \dfrac{\nu}{1 - \nu + \frac{\nu}{q-1}}$, then $P_{Z|X}$ is a degraded version of q-SC(δ).

Remarks:

- Prop follows from computing extremal δ such that $W_\delta^{-1} P_{Z|X}$ is row stochastic.
- Bound on δ can be improved when more is known about $P_{Z|X}$:
  - Markov chain [MP18]: $\delta \leq \nu / \left(1 - (q-1)\nu + \frac{\nu}{q-1}\right)$.
  - Additive noise channel on Abelian group $\mathcal{X}$ [MP18]: $\delta \leq (q-1)\nu$.
  - Alternative bounds for Markov chains [MOS13].
- Many applications in information theory, statistics, and probability [MP18], [MOS13].

Proof Idea: Degradation by Symmetric Channels

Theorem (Effective Input Alphabet Bound)

For any entry-wise strictly positive channel $P_{Z|X} > 0$:

$$C_{\text{perm}}(P_{Z|X}) \leq \frac{\text{ext}(P_{Z|X}) - 1}{2}.$$

Proof Sketch:

- *Degradation by symmetric channels* + *tensorization* of degradation + data processing

$$\Rightarrow \quad I(X_1^n; Y_1^n) \leq I(X_1^n; \tilde{Y}_1^n)$$

where Y_1^n and $\tilde{Y}_1^n$ are outputs of permutation channels with $P_{Z|X}$ and q-SC(δ), resp.

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- *Fano argument* of output alphabet bound $\Rightarrow$ effective input alphabet bound.

Outline

Strictly Positive and "Full Rank" Channels

Achievability and converse bounds yield:

Theorem (Strictly Positive and "Full Rank" Channels)

For any entry-wise *strictly positive* channel $P_{Z|X} > 0$ that is *"full rank"* in the sense that $r \triangleq \text{rank}(P_{Z|X}) = \min\{\text{ext}(P_{Z|X}), |\mathcal{Y}|\}$:

$$C_{\text{perm}}(P_{Z|X}) = \frac{r-1}{2}\,.$$

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Recall Example: C_{perm} of non-trivial binary symmetric channel is $\frac{1}{2}$.

Main Result:
For any entry-wise *strictly positive* channel $P_{Z|X} > 0$:

$$\frac{\operatorname{rank}(P_{Z|X}) - 1}{2} \leq C_{\text{perm}}(P_{Z|X}) \leq \frac{\min\{\operatorname{ext}(P_{Z|X}), |\mathcal{Y}|\} - 1}{2}.$$

Conclusion

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Future Directions:

- Characterize C_{perm} of all (entry-wise strictly positive) channels.

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- Perform finite blocklength analysis (i.e., exact asymptotics for maximum achievable $|\mathcal{M}|$).
- Analyze permutation channels with more complex probability models in the random permutation block.

References

This talk was based on:

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- A. Makur, "**Bounds on permutation channel capacity**," in *Proceedings of the IEEE International Symposium on Information Theory* (ISIT), Los Angeles, CA, USA, June 21-26 2020.
- A. Makur, "**Coding theorems for noisy permutation channels**," accepted to *IEEE Transactions on Information Theory*, July 2020.

Thank You!