Romberg Integration.

Richardson extrapolation revisited.

$$\int_a^b f(x)\,dx = \underbrace{\frac{h}{2}\left[f_0 + 2f_1 + \dots + f_n\right]}_{T_n} + Ch^2 + O(h^4) \qquad \text{— (1)}$$

We can also define $T_{\frac{h}{2}}$ similarly.

$$\text{Error in } T_{\frac{h}{2}} \approx \frac{Ch^2}{4} + O(h^4) \qquad \text{— (2)}$$

From (1) and (2), we can derive

$$\int_a^b f(x)\,dx = \frac{4}{3}T_{\frac{h}{2}} - \frac{1}{3}T_h + O(h^4)$$

$$= \frac{h}{3}\left[f_0 + 2f_{1/2} + 2f_1 + \dots + 2f_{n-\frac{1}{2}} + f_n\right]$$

$$- \frac{h}{6}\left[f_0 + 2f_1 + \dots + 2f_{n-1} + f_n\right] + O(h^4)$$

$$= \frac{h}{6}\left[f_0 + 4f_{\frac{1}{2}} + 2f_1 + \dots + 4f_{n-\frac{1}{2}} + f_n\right] + O(h^4)$$

This is composite Simpson's rule!
We can do this repeatedly - alternate powers are missing in Trapezoid rule - This is called Romberg Integration

Dealing with Singularities.

If a function has a singularity at c, we can write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If a function goes to infinity at some point –

eg. $\displaystyle\int_a^b \frac{g(x)}{(x-a)^\theta}\,dx \qquad 0<\theta<1$

Write this as

$$\int_a^{a+\delta} \frac{g(x)}{(x-a)^\theta}\,dx + \underbrace{\int_{a+\delta}^b \frac{g(x)}{(x-a)^\theta}\,dx}$$

Can be computed using methods we have seen.

Expand g in its Taylor series.

$$g(x) = g(x) + (x-a)g'(a) + \frac{(x-a)^2}{2} g''(a) + \ldots$$

we can then write.

$$\int_a^{a+\delta} \frac{g(x)}{(x-a)^{\theta}}\,dx = \int_a^{a+\delta} \Big(\frac{g(a)}{(x-a)^{\theta}} + g'(a)(x-a)^{1-\theta} + \frac{g''(a)}{2}(x-a)^{(2-\theta)} + \cdots \Big)\,dx$$

$$= g(a)\,\frac{(x-a)^{1-\theta}}{1-\theta}\Big|_a^{a+\delta} + g'(a)\,\frac{(x-a)^{2-\theta}}{2-\theta}\Big|_a^{a+\delta}$$

$$+ \cdots$$

$$= g(a)\,\frac{\delta^{1-\theta}}{1-\theta} + g'(a)\,\frac{\delta^{2-\theta}}{2-\theta} + \frac{g''(a)}{2}\,\frac{\delta^{3-\theta}}{3-\theta} + \cdots$$

This sum can be evaluated (since we know how to compute derivatives and suitably terminated.

What if the upper endpoint of integration is infinite ?

$$\int_a^\infty f(x)\,dx = \int_a^R f(x)\,dx + \int_R^\infty f(x)\,dx.$$

It is often possible to show that for sufficiently large R, the second integral tends to 0.

Alternately write

$$\xi = \frac{1}{x}$$

The second integral becomes

$$\int_0^{1/R} f(1/\xi)\,\xi^{-2}\,d\xi$$

If $f(1/\xi)\,\xi^2$ approaches a finite limit as $\xi \to 0$, we can use standard quadrature schemes.

If this approaches infinity, we fall back to previous case.

$$f(1/\xi)\,\xi^2 \to g(\xi)/\xi^\theta.$$