Solving ODEs.

Midpoint Method.

$$y_{k+\frac{1}{2}} = y_k + \frac{h}{2} f(t_k, y_k)$$

$$y_{k+1} = y_k + h f(t_{k+\frac{1}{2}}, y_{k+\frac{1}{2}}).$$

(Two function evaluations per step).

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local truncation error.

$$y(t_{k+1}) = y(t_{k+\frac{1}{2}}) + \frac{h}{2} f(t_{k+\frac{1}{2}}, y(t_{k+\frac{1}{2}})) + \frac{h^2/2^2}{2} y''(t_{k+\frac{1}{2}}) + O(h^3) \quad \text{— (1)}$$

$$y(t_k) = y(t_{k+\frac{1}{2}}) - \frac{h}{2} f(t_{k+\frac{1}{2}}, y(t_{k+\frac{1}{2}})) + \frac{(h/2)^2}{2} y''(t_{k+\frac{1}{2}}) + O(h^3). \quad \text{— (2)}$$

(1) − (2) gives

$$y(t_{k+1}) - y(t_k) = h f(t_{k+\frac{1}{2}}, y(t_{k+\frac{1}{2}})) + O(h^3). \quad \text{— (3)}$$

Expanding $y(t_{k+\frac{1}{2}})$ about t_k, we have

$$y(t_{k+\frac{1}{2}}) = y(t_k) + \frac{h}{2} f(t_k, y(t_k)) + O(h^2) \quad \text{— (4)}$$

Substituting from (4) into (3), ~~ye~~ we get

$$y(t_{k+1}) - y(t_k) = hf\left(t_{k+\frac{1}{2}},\ y(t_k) + \frac{h}{2} f(t_k, y(t_k)) + O(h^2)\right)$$
$$+ O(h^3)$$

$$= hf\left(t_{k+\frac{1}{2}},\ y(t_k) + \frac{h}{2} f(t_k, y(t_k))\right) + O(h^3)$$

(this follows from assumption on Lipschitz continuity)

$$hf(t, y + O(h^2)) = hf(t, y) + O(h^3)$$

$$\therefore \quad y_{k+1} = y_k + hf\left(t_{k+\frac{1}{2}},\ y_k + \frac{h}{2} f(t_k, y_k)\right)$$

or $$\frac{y_{k+1} - y_k}{h} = f\left(t_{k+\frac{1}{2}},\ y_k + \frac{h}{2} f(t_k, y_k)\right).$$

local truncation error is $O(h^2)$.

(Second order accurate).

Methods Based on Quadrature Formulae.

$$y'(t) = f(t, y(t))$$

$$y(t_0) = y_0.$$

Integrating from t to $t+h$ on both sides, we get

$$y(t+h) - y(t) = \int_t^{t+h} f(s, y(s))\, ds.$$

The integral on the right can be evaluated using quadrature formulae.

→ Using trapezoid rule:

$$\int_t^{t+h} f(s, y(s))\, ds = \frac{h}{2}\left[f(t, y(t)) + f(t+h, y(t+h))\right] + O(h^3)$$

∴ we get the formula

$$y_{k+1} = y_k + \frac{h}{2}\left[f(t_k, y_k) + f(t_{k+1}, y_{k+1})\right]$$

Implicit method! y_{k+1} is on both sides, so you have to solve a nonlinear equation at each step.

Trapezoid Rule in ODE Solvers.

- local truncation error is $O(h^2)$ but

  Solving equations at each timestep is expensive.

Heun's Method:

Estimate y_{k+1} using Euler's method and

use this in the RHS of prev. equation.

$$\tilde{y}_{k+1} = y_k + h f(t_k, y_k)$$

$$y_{k+1} = y_k + \frac{h}{2}\left[f(t_k, y_k) + f(t_{k+1}, \tilde{y}_{k+1}) \right].$$

Runge-kutta Methods.

Heun's method is also called Second Order Runge kutta method.

Alternate derivation.

$$\tilde{y}_{k+\alpha} = y_k + \alpha h f(t_k, y_k)$$

$$y_{k+1} = y_k + \beta h f(t_k, y_k) + \gamma h f(t_k + \alpha h, \tilde{y}_{k+\alpha})$$

Parameters α, β, γ are selected to cancel as many terms of the Taylor approximation as possible.

By selecting

$$\cancel{\alpha +}\ \beta + \gamma = 1$$

$$\alpha\gamma = \frac{1}{2}$$

we can cancel h and h^2 term.

Methods of this form are called Runge-kutta methods.

Heun's method uses $\alpha = 1$ $\beta = \gamma = \frac{1}{2}$.

local truncation error $= O(h^2)$.

Runge Kutta Methods.

We can introduce two intermediate values and we can achieve 4^{th} order accuracy.

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Classical Runge Kutta Method.

$$q_1 = f(t_k, y_k)$$

$$q_2 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} q_1\right).$$

$$q_3 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} q_2\right).$$

$$q_4 = f(t_k + h, y_k + h q_3).$$

$$y_{k+1} = y_k + \frac{h}{6}\left[q_1 + 2q_2 + 2q_3 + q_4\right].$$

If f is independent of y. This is the same as Simpson's rule.

$$y(t+h) - y(t) = \int_t^{t+h} f(s)\,ds \approx \frac{h}{6}\left[f(t) + 4f\left(t+\frac{h}{2}\right) + f(t+h)\right]$$

Simpsons rule is $O(h^5)$ $\therefore$ this is $O(h^4)$.

Runge-Kutta-Fehlberg Method.

We want to estimate error and adaptively change step size.

By carefully choosing parameters, Fehlberg showed that one needs only 6 function evaluations for a fifth-order accurate method, with an associated fourth order method used to estimate error and vary step size accordingly.

Some more definitions.

One step method: $y_{k+1} = y_k + h\,\psi(t_k, y_k, h)$.

Defn. A one-step method is consistent if

$$\lim_{h\to 0} \psi(t, y, h) = f(t, y).$$

or

$$\lim_{h\to 0} \left[\frac{y_{k+1} - y_k}{h} - \psi(t_k, y(t_k), h)\right] = 0$$

Since $\lim_{h\to 0} \dfrac{y_{k+1} - y_k}{h} = f(t_k, y(t_k))$.

All one step methods above are consistent.

Defn. A one-step method is stable if there is a constant k and step size $h_0 > 0$ such that the difference between two solutions y_n and $\tilde{y}_n$ with initial values y_0 and $\tilde{y}_0$, respectively, satisfies

$$|y_n - \tilde{y}_n| \leq k\,|y_0 - \tilde{y}_0|.$$

whenever $h \leq h_0$ and $nh \leq T - t_0$.

If $\psi(t, y, h)$ is Lipschitz continuous in y,
(There exists a constant L such that).

$$|\psi(t, y, h) - \psi(t, \tilde{y}, h)| \leq L|y - \tilde{y}|$$

Then

$$y_{k+1} = y_k + h\,\psi(t_k, y_k, h) \text{ is stable.}$$

Multistep Methods : Adams-Bashforth and Adams-Moulton Methods.

$$y(t_{k+1}) = y(t) + \int_{t_k}^{t_{k+1}} f(s, y(s))\, ds.$$

Use computed values $y_k, y_{k-1}, \dots y_{k-m+1}$
(prev m values).

We can fit $P_{m-1}(s)$ to be the $(m-1)$ degree interpolant for $f(t_k, y_k), f(t_{k-1}, y_{k-1}), \dots f(t_{k-m+1}, y_{k-m+1})$.

The Lagrange form of the interpolant is

$$P_{m-1}(s) = \sum_{l=0}^{m-1} \left(\prod_{\substack{j=0 \\ j \neq l}}^{m-1} \frac{s - t_{k-j}}{t_{k-l} - t_{k-j}} \right) f(t_{k-l}, y_{k-l})$$

Replace $f(s, y(s))$ in the integral by $P_{m-1}(s)$.

$\therefore$

$$y_{k+1} = y_k + \int_{t_k}^{t_{k+1}} P_{m-1}(s)\, ds$$

$$= y_k + h \sum_{\ell=0}^{m-1} b_\ell f(t_{k-\ell}, y_{k-\ell})$$

$$b_\ell = \frac{1}{h} \int_{t_k}^{t_{k+1}} \left(\prod_{\substack{j=0 \\ j \neq \ell}}^{m-1} \frac{s - t_{k-j}}{t_{k-\ell} - t_{k-j}} \right) ds.$$

This is the m-step Adams Bashforth method.

To start the process, $y_0 \ldots y_m$ are computed using one-step methods.

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$m=1$ gives the Euler method.

$m=2$: $y_{k+1} = y_k + h \left[b_0 f(t_k, y_k) + b_1 f(t_{k-1}, y_{k-1}) \right]$

$b_0 = \frac{3}{2}$ $b_1 = -\frac{1}{2}$.

$$y_{k+1} = y_k + h\left[\frac{3}{2} f(t_k, y_k) - \frac{1}{2} f(t_{k-1}, y_{k-1})\right].$$

The error in approximating the integral is $O(h^{m+1})$

$\therefore$ local truncation error is $O(h^m)$.

Adams-Moulton methods are explicit.

Use polynomial q_m to interpolate

$$f(t_{k+1}, y_{k+1}),\ f(t_k, y_k) \ldots,\ f(t_{k-m+1}, y_{k-m+1}).$$

Use q_m instead of f in the integral.

$$y_{k+1} = y_k + h \sum_{l=0}^{m} c_l f(t_{k+1-l}, y_{k+1-l})$$

$$c_l = \frac{1}{h} \int_{t_k}^{t_{k+1}} \left(\prod_{\substack{j=0 \\ j \neq l}}^{m} \frac{s - t_{k+1-j}}{t_{k+1-l} - t_{k+1-j}} \right) ds.$$

For $m=0$, we set

$$y_{k+1} = y_k + h f(t_{k+1}, y_{k+1}).$$

Backward Euler method

$m=1$

$$y_{k+1} = y_k + \frac{h}{2}[f(t_{k+1}, y_{k+1}) + f(t_k, y_k)]$$

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The local truncation error of Adams-Moulton method is $O(h^{m+1})$.