Chapter 3. Monte Carlo Methods.

Simulating phenomena based on probabilities.

Basic Statistics.

<u>Random Variables</u>: Function that maps outcomes of an experiment to the set of real numbers.

Examples:

X: set of heads in 8 flips of a fair coin

Y: total value on three rolls of a fair die.

<u>Discrete Random Variables</u>: Random variables that take only discrete values.

Examples: X and Y above.

(Can you come up with an example of a continuous random variable?

Why is this distinction important?
)

Discrete Random Variables

Say random variable X takes values $x_1, x_2 \dots x_n$.

With each value x_i, we associate probability p_i

Example:

X: total value in two rolls of a fair dice.

$X \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Say $x_i = 6$

What is p_i ?

$x_i = 6$ can happen as $(1,5), (2,4), (3,3),$
$(4,2), (5,1)$.

There are 5 out of a possible 36 outcomes.

$\therefore p_i = \frac{5}{36}$

$$\sum_{i=1}^{m} p_i = 1$$

(m is all possible values of the random variable).

Expected value:

$$E(x) = \sum_{i=1}^{m} p_i x_i \quad (\text{also written as } \mu).$$

Example: Expected value of the roll of a single dice.

$$E(x) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \quad \cdots \quad + \frac{1}{6} \times 6 = \frac{7}{2}$$

If all values of the r.v. are equally likely and there are n possible values, we have $p_i = \frac{1}{n}$

and $E(x) = \sum_{i=1}^{n} \frac{1}{n} x_i = \frac{1}{n} \sum_{i=1}^{n} x_i$

Expected value of a function of a random variable.

Let $g(x)$ be a function of r.v. X

$$E(g(x)) = \sum_{i=1}^{m} p_i \, g(x_i)$$

Example: Say $g(x) = aX + b$

$$E(g(x) = \sum_{i=1}^{m} p_i \, (a x_i) + b = a \sum_{i=1}^{m} p_i x_i + b$$

$$= a E(x) + b.$$

Example: Say $g(x) = x^2$

$$E(g(x)) = \sum_{i=1}^{m} p_i x_i^2$$

Variance of a Random Variable

$$\mathrm{Var}(X) = E\left((X-\mu)^2\right) = \sum_{i=1}^{m} p_i (x_i-\mu)^2$$

$$= \sum_{i=1}^{m} p_i \left(x_i^2 - 2x_i\mu + \mu^2\right)$$

$$= \sum_{i=1}^{m} p_i x_i^2 - 2\mu \sum_{i=1}^{m} p_i x_i + \mu^2$$

$$= E(X^2) - 2(E(X))^2 + E(X)^2$$

$$= E(X^2) - (E(X))^2$$

Variance of $c.X$ (c is a constant) is

$$c^2.\ \mathrm{Var}(X).$$

$\sqrt{\mathrm{Var}(X)}$ = Standard deviation of X or $\sigma(X)$.

Example. Variance of a single roll of a fair die.

$$\mathrm{Var}(X) = \underbrace{\frac{91}{6}}_{E(X^2)} - \left(\frac{7}{2}\right)^2 \approx \frac{35}{12}.$$

$$\sigma(X) = \sqrt{\frac{35}{12}} = 1.708.$$

More on Expectation and variance.

Say X & Y are two r.v.s.

$E(X+Y) = E(X) + E(Y)$

Proof: $E(X+Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i + y_j) \, p(X = x_i \wedge Y = y_j)$

$= \sum_{i=1}^{m} x_i \sum_{j=1}^{n} p((X = x_i) \wedge (Y = y_j))$

$+ \sum_{j=1}^{n} y_j \sum_{i=1}^{m} p(X = x_i \wedge Y = y_j)$

$= \sum_{i=1}^{m} x_i p_i + \sum_{j=1}^{n} y_j q_j = E(X) + E(Y)$

What is variance of $c_1 X + c_2 Y$

$Var(c_1 X + c_2 Y) = E((c_1 X + c_2 Y)^2) - (E(c_1 X + c_2 Y))^2$

$= E(c_1^2 X^2 + 2c_1 c_2 XY + c_2^2 Y^2) - (c_1 E(X) + c_2 E(Y))^2$

$= c_1^2 E(X^2) + 2c_1 c_2 E(XY) + c_2^2 E(Y^2) - c_1^2 (E(X))^2$
$- 2c_1 c_2 E(X) E(Y) - c_2^2 (E(Y))^2$

$= c_1^2 Var(X) + c_2^2 Var(Y) + 2c_1 c_2 (E(XY) - E(X)E(Y))$

We define covariance of X & Y as

$Cov(X, Y) = E(XY) - E(X)E(Y)$

<u>Independent r.v.s</u> $\quad p(X = x_i \wedge Y = y_j) = p(X = x_i) \cdot p(Y = y_j)$

Two flips of a fair coin.

Example. $X = 1$ if first coin is heads, 0 otherwise.

$Y = 1$ if second coin is heads, 0 otherwise.

Are X & Y independent?

$$P(X = 1 \wedge Y = 0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(X=1) \cdot P(Y=0).$$

Similarly for other three possible outcomes.

in each case, the independence

____.

If X & Y are independent, $\mathrm{Cov}(X,Y) = 0$

and $\mathrm{Var}(c_1 X + c_2 Y) = c_1^2 \mathrm{Var}(X) + c_2^2 \mathrm{Var}(Y)$.

Continuous Random Variables.

a random variable X that can take a continuous value.

Why is this different?

Say you are generating a random integer in the range 1.. 100

What is $p(55)$? $\frac{1}{100}$

Say you are now generating real valued numbers (in infinite precision) in the range 1.. 100

What is $p(55)$? 0 !

$\therefore$ we must define the cumulative distribution function.

$$F(x) = \text{Prob}(X < x)$$

and Probability ~~distrib~~ density function

$$\rho(x)\,dx = \text{Prob}(X \in [x, x + dx])$$

$$\equiv F(x+dx) - F(x) \qquad dx \to 0.$$

$$\rho(x) = F'(x) \quad \text{w}$$

$$F(x) = \int_{-\infty}^{x} \rho(t)\,dt.$$

$$E(x) = \int_{-\infty}^{\infty} x\,\rho(x)\,dx$$

$$\int_{-\infty}^{\infty} \rho(x) = 1$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \rho(x)\,dx$$

and $\mathrm{Var}(x) = E(x^2) - (E(x))^2$

Example.

Uniform dist in $[0,1]$

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1 \end{cases} \qquad \rho(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \end{cases}$$

$$E(x) = \int_{-\infty}^{\infty} x\rho(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}.$$

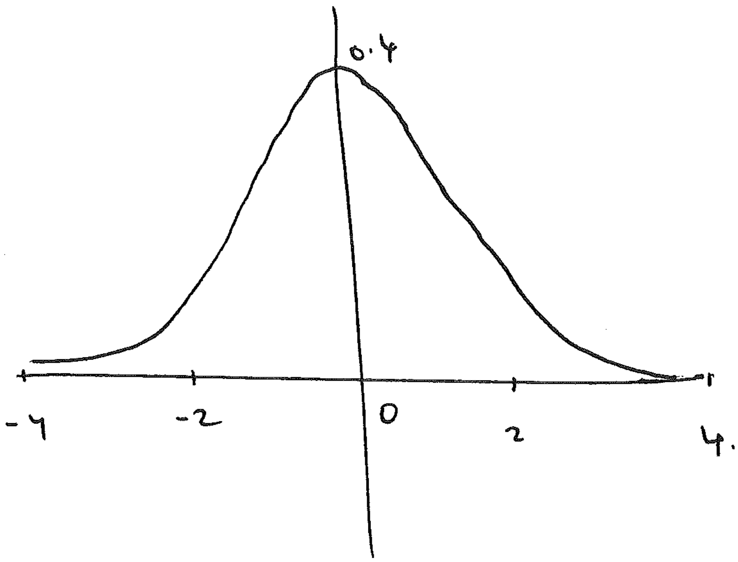
$$\mathrm{Var}(x) = \int_0^1 x^2 dx - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Normal (Gaussian) Distribution.

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

μ : mean

σ^2 : variance.



$\mu = 0$

$\sigma = 1$