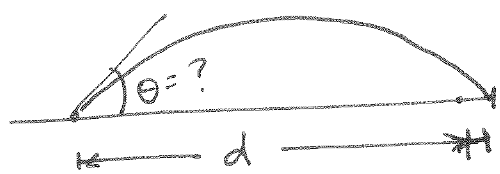
# Chapter 4.

## Solution of a Single Nonlinear Equation in One Unknown

Example.




what angle should you fire a projectile to hit a target d mts away?

Initial muzzle velocity $= V_0$

Height $y(t)$ is given by

$$y''(t) = -g \quad ; \quad y'(t) = -gt + c$$

$$y(t) = -\frac{1}{2}gt^2 + c_1 t + c_2$$

Initial Conditions : $y(0) = 0$

$y'(0) = V_0 \sin\theta$

$$\therefore \quad y(t) = -\frac{1}{2}gt^2 + V_0 \sin\theta \, t$$

Time it comes back to ground $(y=0)$ is given by

$$0 = -\frac{1}{2}gt^2 + V_0 \sin\theta \, t$$

$$t = 0 \quad \text{or} \quad t = \frac{2V_0 \sin\theta}{g}$$

Distance traveled in this time is

$$V_0 \cos\theta \cdot t = \frac{2 V_0^2 \cos\theta \sin\theta}{g}$$

To find the angle $\theta$, we must now solve

$$\frac{2 V_0^2 \cos\theta \sin\theta}{g} = d$$

One must consider many factors:

→ Air resistance

→ No solution (target out of range)

→ Multiple solutions

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While in some cases, analytical solutions may exist, we may need to compute the solution on a computer.

# Solving Nonlinear equations in One variable.

## Bisection Method.

Solve the cannon angle problem using trial and error!

Fire once - if you overshoot, lower the cannon. keep lowering until you undershoot. Now you have two angles - one at which you overshoot and one at which you undershoot.

→ what do you think you should do?

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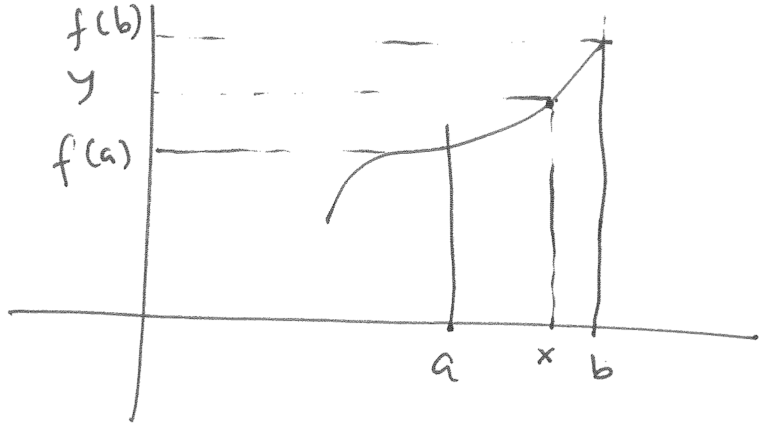
Say you are trying to solve

$f(x) = 0$, $f$ is a continuous function

and you have points $x_1$ and $x_2$ such that the sign of $f(x)$ changes between these points. Then there must exist $x'$ between $x^1$ & $x^2$ such that $f(x') = 0$

Intermediate Value Theorem:

If $f$ is continuous on $[a,b]$ and $y$ lies between $f(a)$ and $f(b)$ then there is a point $x \in [a,b]$ such that $f(x) = y$




If one of $f(a)$ or $f(b)$ is positive and the other is negative, then the solution $f(x)=0$ must belong in this interval with $x \in [a,b]$

Bisection method. Starting from interval $[a,b]$ as defined, compute $f\left(\frac{a+b}{2}\right)$. Replace one of the end points ($f(a)$ or $f(b)$) with $f\left(\frac{a+b}{2}\right)$ so that the sign inversion is still satisfied by the new interval.

Bisection Method. 

Example: find a root of $f(x) = x^3 - x^2 - 1$

in the interval $[0, 2]$

First check for sign inversion:

$f(0)$ is negative.

$f(2)$ is positive

| n (iteration) | $a_i$ | $b_i$ | midpoint $r_i$ | $f(r_i)$ |
|---|---|---|---|---|
| 0 | 0 | 2 | 1 | -1 |
| 1 | 1 | 2 | 1.5 | 0.125 |
| 2 | 1 | 1.5 | 1.25 | -0.609 |
| 3 | 1.25 | 1.5 | 1.375 | -0.291 |
| | ⋮ | | | |
| 8 | 1.453125 | 1.46875 | 1.4609375 | -0.016 |

Rate of Convergence.

Interval is halved in each step.

After $k$ steps, the interval is $\frac{b-a}{2^k}$

If we want a solution to within $2\delta$, we have

$$\frac{|b-a|}{2^k} \leq 2\delta \; , \quad 2^{k+1} \geq \frac{|b-a|}{\delta}$$

$$\text{or} \quad k \geq \log_2\left(\frac{|b-a|}{\delta}\right) - 1$$

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Notes. - Each step reduces error by a constant factor

∴ Bisection method is said to converge linearly.

- One can think of optimizations that do not take the midpoint, but rather find intermediate point by linear interpolation between endpoints. In this case the convergence rate is not guaranteed.

- Bisection method is guaranteed to converge, however, finding initial interval may be hard.

Taylor's Theorem. (with remainder)

Thm : let $f, f', f'', \ldots f^{(n)}$ be continuous in $[a,b]$ and let $f^{(n+1)}(x)$ exist for all $x \in (a,b)$. Then there is a number $\xi \in (a,b)$ such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^n}{n!}f^{(n)}(a) + \frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi)$$

writing $a = x$ and $b = x+h$, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\xi)$$

where $\xi \in (x, x+h)$

If the expansion is around a fixed point $a$, we can write

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{n+1}(\xi)$$

The remainder term in Taylor's Series about point a is

$$R_n(x) = f(x) - \sum_{j=0}^{n} \frac{(x-a)^j}{j!} f^{(j)}(a)$$

$$\approx \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

this is also written as $O((x-a)^{n+1})$

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Depending on properties of f, the Taylor Series may converge to f everywhere or in some interval around a, or only at point a itself.

For functions such as $e^x$, $\sin x$ and $\cos x$, Taylor Series converges everywhere.

Taylor series around $\emptyset$ is called Maclaurin Series.

Example

Find Taylor Series expansion of $e^x$ about 1

$f(1) = e^1$

$f'(1) = e^1$

$f''(1) = e^1$

$\vdots$

$$\therefore \quad e^x = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \cdots$$

$$= e^1\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right)$$

$$= e^1 \sum_{k=0}^{\infty} \frac{(x-1)^k}{k!}$$

Example: Find Maclaurin Series expansion for $\cos x$

$f(0) = \cos 0 = 1 \qquad f'(0) = -\sin(0) = 0$

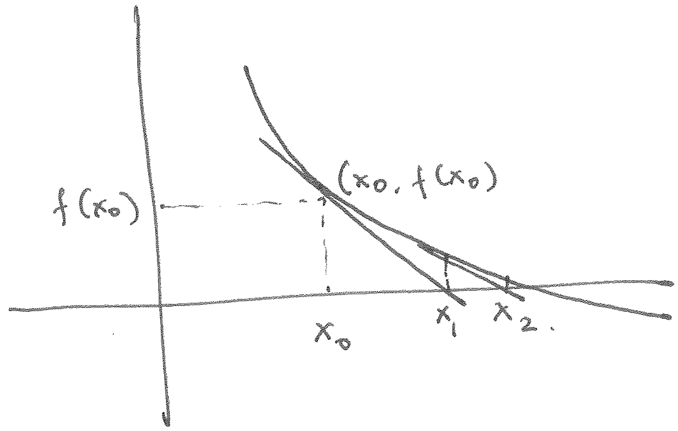
$f''(0) = -\cos 0 = -1 \qquad f'''(0) = \sin(0) = 0$

$f^{(4)}(0) = \cos 0 = 1$

$$\therefore \quad \cos(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Newton's Method. 




Start at point $x_0$, draw a tangent to function $f$ at $x_0$. The next guess is where the tangent intersects the $x$ axis.

We draw tangent at point $(x_0, f(x_0))$

Slope of the line is $f'(x_0)$

$\therefore$ equation of the line is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

To intersect this line with $x$ axis, we set $y = 0$

$$\therefore \quad f'(x)(x - x_0) = -f(x_0)$$

$$\text{or} \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}$$ (Assuming $f'(x_0) \neq 0$)

Newton's Method.

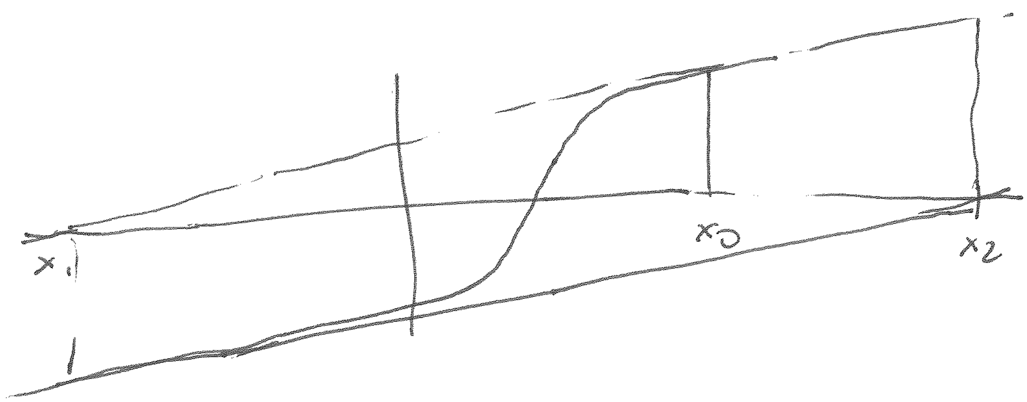
for ~~x~~ k = 0, 1, 2 . . .

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

The recurrence was first given by Joseph Raphson.

For this reason, it is also sometimes called Newton-Raphson.

→ Newtons Method may not always converge.




We can combine Newton with bisection to guarantee convergence. If a Newton step takes you outside a bisection interval, we reject the Newton iterate and use a bisection iterate.

This guarantees convergence.

Deriving Newton's Method using Taylor's Thm.

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(\xi)$$

for some $\xi$ between $x_0$ and $x$

if $f(x') = 0$, we have

$$0 = f(x_0) + (x' - x_0) f'(x_0) + \frac{(x'-x_0)^2}{2!} f''(\xi)$$

Now, ignoring the residual term, we get an approximation to the solution $x_1 \approx x'$ and we have

$$0 = f(x_0) + (x_1 - x_0) f'(x_0)$$

$$\text{or } x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Convergence.

Thm: If $f \in C^2$, if $x_0$ is sufficiently close to root $x'$ and if $f'(x_0) \neq 0$, then Newton's method converges to $x'$, and ultimately the convergence rate is quadratic; i.e, there exists constant $C' = \left| \frac{f''(x')}{2f'(x')} \right|$ such that

$$\lim_{k \to \infty} \frac{|x_{k+1} - x'|}{|x_k - x'|^2} = C'$$

# Convergence Rate of Newton's Method.

Proof.

Consider the $k^{th}$ iterate. Writing the Taylor Series with the residual term, we have

$$x' = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{(x'-x_k)^2}{2}\frac{f''(\xi_k)}{f'(x_k)} \quad — (1)$$

This simply follows from

$$\boxed{0 = f(x_k) + (x'-x_k)f'(x_k) + \frac{(x'-x_k)^2}{2}f''(\xi)}$$

from previous page with $x_k$ replacing $x_0$

$\therefore$

We also know that

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad — (2)$$

(2) − (1) gives us

$$x_{k+1} - x' = (x_k - x')^2 \cdot \frac{f''(\xi_k)}{2f'(x_k)}$$

Proof (contd).

$f''$ is continuous and $f'(x') \neq 0$

$$C' = \left| \frac{f''(x')}{2f'(x')} \right|$$

then for any $c > c'$, there is an interval about $x'$ in which $\left| \frac{f''(\xi)}{2f'(x)} \right| \leq c$ for any $x$ and $\xi$ in this interval.

If for some $k$, $x_k$ lies in this interval and if

$|x_k - x'| < \frac{1}{c}$ (if $x_0$ is close to $x'$, this also holds for $k=0$).

we have

$$|x_{k+1} - x'| \leq c\,|x_k - x'|^2 < |x_k - x'|$$

This implies that $x_{k+1}$ also lies in the interval and satisfies $|x_{k+1} - x'| < \frac{1}{c}$

$$\begin{aligned}
|x_{k+1} - x'| &\leq c\,|x_k - x'|^2 \\
&\leq (c\,|x_k - x'|).\,|x_k - x'| \\
&\leq (c\,|x_k - x'|).\,c\,(|x_{k-1} - x'|)\,(x_{k-1} - x') \\
&\;\;\vdots \\
&\leq c\,|x_k - x'|^{k+1-k}\,|x_k - x'|
\end{aligned}$$

Since $C|x_k - x'| < 1$ it follows that (6)

$$C \, |x_k - x'|^{k+1-k} \to 0 \quad \text{as } k \to \infty$$

$$\Rightarrow \quad x_k \to x' \text{ as } k \to \infty$$

Also since $x_k$ and therefore $\xi_k \to x'$ it follows that

$$\frac{|x_{k+1} - x'|}{|x_k - x'|^2} = \left| \frac{f''(\xi_k)}{2f'(x_k)} \right| \to C'.$$

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We assumed that $\left| \frac{f''}{2f'} \right| \le C$ and $|x_0 - x'| < \frac{1}{C}$

This dictates that our starting iterate must be sufficiently close to solution $x'$

Example

Use Newton's method to compute $\sqrt{2}$

$$f(x) = x^2 - 2 = 0$$

$$f'(x) = 2x$$

$$\therefore \quad x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k} \qquad k = 0, 1, \ldots$$

Start at $x_0 = 2$, error $e_k = x_k - \sqrt{2}$

$x_0 = 2$ $\qquad e_0 = 0.59$

$x_1 = 1.5$ $\qquad e_1 = 0.086$

$x_2 = 1.4167$ $\qquad e_2 = 0.0025$

$x_3 = 1.4142157$ $\qquad e_3 = 2.1 \times 10^{-6} \approx 0.35\, e_2^2$

The constant $\left| \dfrac{f''(x')}{2f'(x')} \right|$ is $\dfrac{1}{2\sqrt{2}} = 0.3536$.

The iteration fails for $x_0 = 0$

for $x_0 > 0$ it converges to $\sqrt{2}$

for $x_0 < 0$ it converges to $-\sqrt{2}$.