Quasi - Newton Methods

→ Newton's method requires computation of the function and its derivative at each iteration.

→ This may not be easy

→ The function may be complex and not explicit

→ Derivatives may be hard to compute.

Can we approximate the derivative ?

$$x_{k+1} = x_k - \frac{f(x_k)}{g_k}$$

$$g_k \simeq f'(x_k)$$

These methods are called quasi-Newton methods.

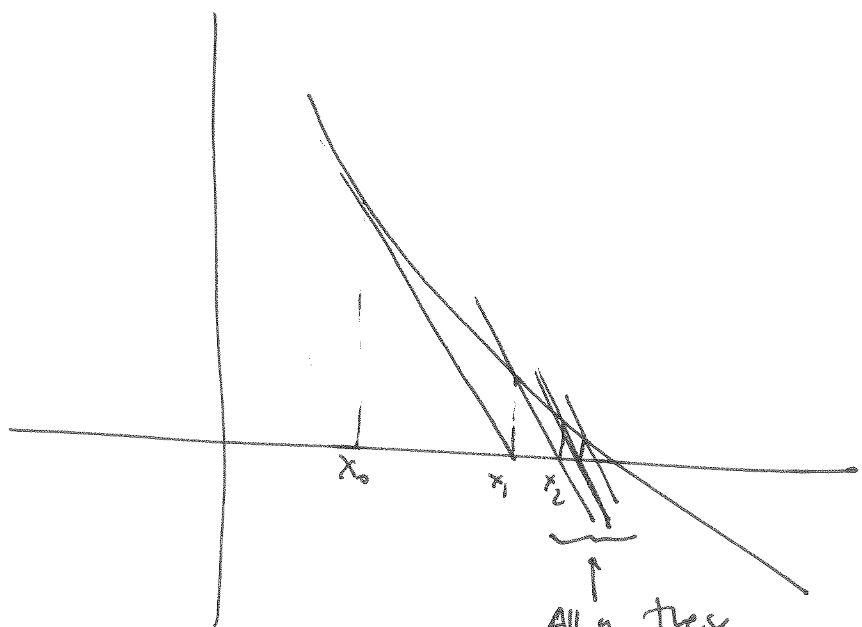
Constant Slope methods.

Compute f' once at x_0 and set

$$g_k = f'(x_0)$$

(use the same slope at each iteration).

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_0)} \quad \text{—— (1)}$$



All of these lines have the same slope $(f'(x_0))$

Analysis: Expand $f(x_k)$ around root x^*

$$f(x_k) = (x_k - x^*) f'(x^*) + O(x_k - x^*)^2$$

Subtracting x^* from both sides of (1), we have

$$e_{k+1} = e_k - \frac{f(x_k)}{f'(x_0)} = e_k \left(1 - \frac{f'(x^*)}{f'(x_0)}\right) + O(e_k^2)$$

Constant Slope method (contd.)

if $\left| 1 - \frac{f'(x^*)}{f'(x_0)} \right| < 1$, then for x_0 sufficiently close to x^*, the method converges, and the convergence is linear.

<u>Optimization</u>: Instead of using $f'(x_0)$ over all iterations, periodically update the derivative as convergence slows down. This poses the trade-off of iteration cost vs. iteration count.

Secant Method.

Here we use

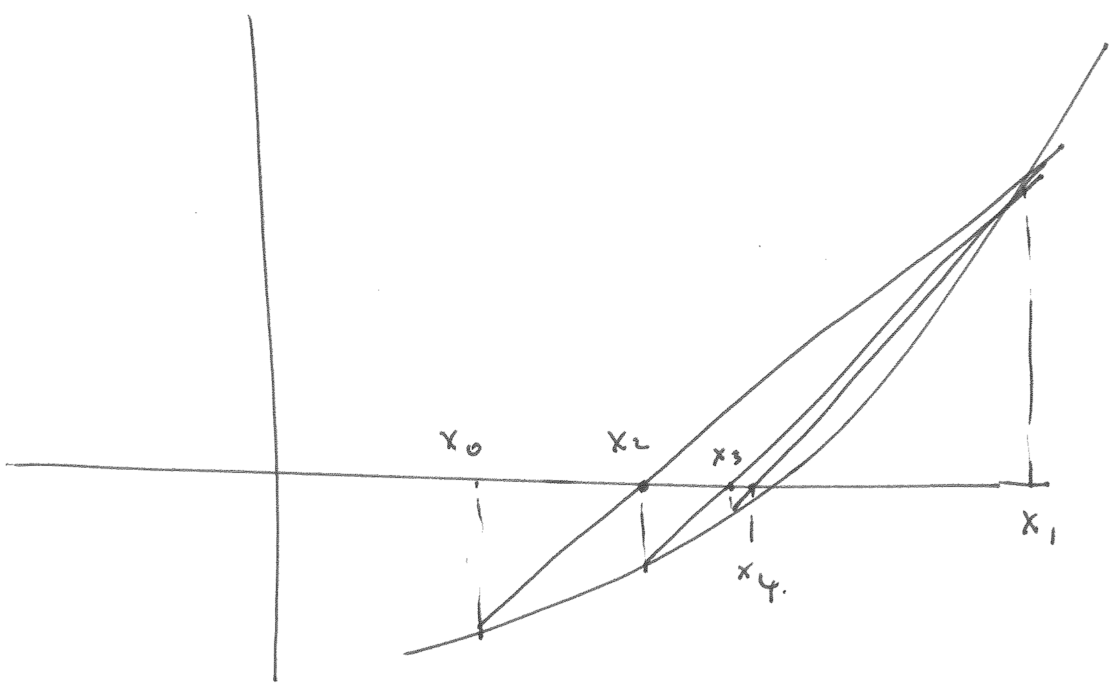
$$g_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

When $x_k \to x_{k-1}$ (close to convergence),

$$g_k \approx f'(x_k)$$

$$x_{k+1} = x_k - \frac{f(x_k)\,(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

Need two points to start.



Analysis

Order of convergence is $\frac{1+\sqrt{5}}{2}$

Lemma. If $f \in C^2$, x_0 and x_1 are sufficiently close to the root x^* and if $f'(x^*) \neq 0$, then

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k e_{k-1}} = C_* \quad \text{— (3)}$$

$$C_* = \frac{f''(x^*)}{2f'(x^*)}$$

Suppose $|e_{k+1}| \simeq a\,|e_k|^{\alpha}$ — (1) for large k.

(we want to show that $\alpha = \frac{1+\sqrt{5}}{2}$.

$$|e_{k-1}| \simeq \left(\frac{|e_k|}{a}\right)^{1/\alpha} \quad \text{— (2)}$$

from (1), (2), and (3), we have

$$a\,|e_k|^{\alpha} \simeq C\,|e_k|\,(|e_k|/a)^{1/\alpha} = C\,|e_k|^{1+\frac{1}{\alpha}}\,a^{-\frac{1}{\alpha}}$$

or $|e_k|^{1+\frac{1}{\alpha}-\alpha} = C'$ (some constant)

or $1 + \frac{1}{\alpha} - \alpha = 0$ or $\alpha = \frac{1 \pm \sqrt{5}}{2}$ (α is +ve)

Proof of lemma.

The Secant iteration is

$$x_{k+1} = x_k - \frac{f(x_k)\,(x_k - x_{k-1})}{(f(x_k) - f(x_{k-1}))}$$

Subtract x^* from both sides to get

$$e_{k+1} = e_k - \frac{f(x_k)\,(e_k - e_{k-1})}{f(x_k) - f(x_{k-1})}$$

$$= \frac{f(x_k).\,e_{k-1} - f(x_{k-1})\,e_k}{f(x_k) - f(x_{k-1})}$$

$$= e_k\, e_{k-1} \left[\frac{f(x_k)/e_k - f(x_{k-1})/e_{k-1}}{x_k - x_{k-1}} \times \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \right]$$

The second factor converges to $f'(x^*)$

The first factor is

$$\frac{f(x_k)/e_k - f(x_{k-1})/e_{k-1}}{x_k - x_{k-1}}$$

This can be approximated as

$$\frac{\dfrac{f(x_k)-f(x^*)}{x_k - x^*} - \dfrac{f(x_{k-1})-f(x^*)}{x_{k-1}-x^*}}{x_k - x_{k-1}} \quad \text{— (3)}$$

Writing $g(x) = \dfrac{f(x)-f(x^*)}{x-x^*}$ — (4)

Notice that expression (3) converges to $g'(x^*)$

as x_k and x_{k-1} approach x^*

Now from (4)

$$g'(x) = \frac{(x-x^*)f'(x) - (f(x)-f(x^*))}{(x-x^*)^2}$$

$$\lim_{x\to x^*} g'(x) = \lim_{x \to x^*} \frac{(x-x^*)\cdot f''(x)}{2(x-x^*)} = \frac{f''(x)}{2}$$

$$\therefore \lim_{k\to\infty} \frac{e_{k+1}}{e_k} = \frac{f''(x^*)}{2} \times \frac{1}{f'(x^*)}$$

Note on Secant method.

Instead of always working with x_{k+1} and x_k,

Start with a pair of points where the root is guaranteed to be in the middle of (like bisection).

Then at each step, take x_{k+1} and one of x_k or x_{k-1} to always guarantee this property as you iterate.

This method is called regula falsi and may only have linear convergence, but is guaranteed to converge.

Fixed Point Iterations.

$$x_{k+1} = \phi(x_k).$$

Many ways to cast $f(x)=0$ into a fixed point iteration.

e.g $\quad x + f(x) = x \qquad (f(x)=0)$

$$x - f(x) = x$$

$$f(x) \equiv x - \phi(x) = 0$$

$$f(x) \equiv \exp(x - \phi(x)) = 1$$

Newton and Constant Slope methods are examples of fixed point iterations.

For Newton's method, $\quad x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$

$$\phi(x) = x - \frac{f(x)}{f'(x)}.$$

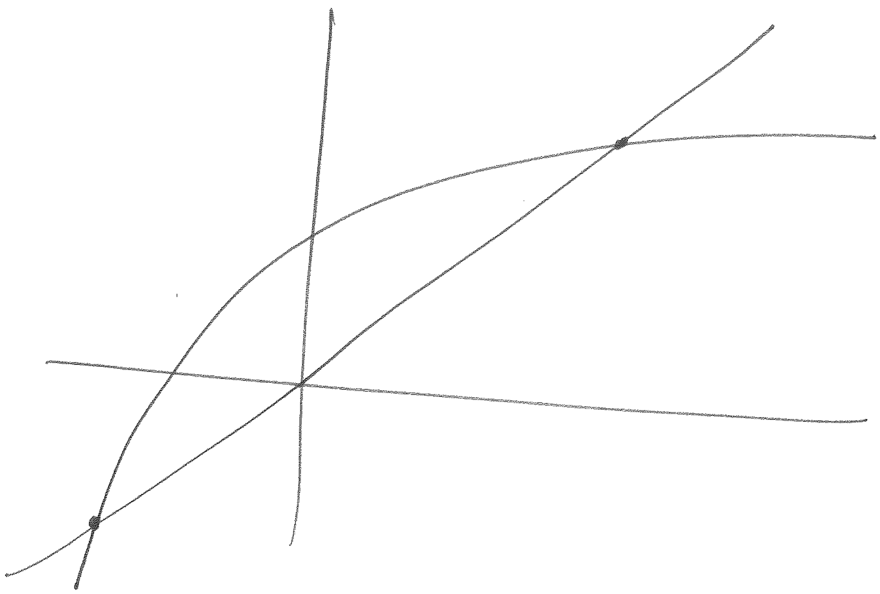
For Constant slope method

$$\phi(x) = x - f(x)/g \qquad (g = f'(x_0)).$$

Geometrically, the Solutions lie at the intersection

of $y = \phi(x)$ and

$y = x$.



Solve $x^2 - x - 6 = 0$

or $x = x^2 - 6$

$\phi(x) = x^2 - 6$.