Chapter 6: Conditioning of problems.

Conditioning: How sensitive the answer is to small changes in input.

f: Scalar-valued function of a scalar argument x

$\hat{x}$ is close to x

· How close is $y = f(x)$ to $\hat{y} = f(\hat{x})$.

→ In an absolute sense

$$|\hat{y} - y| \approx C(x)\,|\hat{x} - x|$$

· $C(x)$ is the absolute condition number of function f at point x

→ In a relative sense

$$\left|\frac{\hat{y} - y}{y}\right| \approx \kappa(x)\left|\frac{\hat{x} - x}{x}\right|$$

$\kappa(x)$ is the relative condition number of f at x

Deriving absolute condition number.

$$\hat{y} - y = f(\hat{x}) - f(x) = \frac{f(\hat{x}) - f(x)}{\hat{x} - x}(\hat{x} - x)$$

if $\hat{x} - x \to 0$ $\qquad \dfrac{f(\hat{x}) - f(x)}{\hat{x} - x} \approx f'(x)$

$$\therefore \quad \hat{y} - y \approx f'(x) \cdot (\hat{x} - x)$$

$$\therefore \quad c(x) \approx |f'(x)|$$

Relative Condition number.

$$\frac{\hat{y} - y}{y} = \frac{f(\hat{x}) - f(x)}{\hat{x} - x} \cdot \frac{\hat{x} - x}{x} \cdot \frac{x}{f(x)}$$

$$\therefore \quad \frac{\hat{y} - y}{y} \approx f'(x) \frac{x}{f(x)} \cdot \frac{\hat{x} - x}{x} \qquad \text{as } \hat{x} - x \to 0$$

$$\therefore \quad K(x) \approx \left| \frac{x f'(x)}{f(x)} \right|$$

Example

$f(x) = 2x$

$f'(x) = 2$

$\therefore \quad C(x) = 2$

$K(x) = \frac{2x}{2x} = 1$

This is well conditined in both absolute and relative sense for all x

Example. $f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2} x^{-1/2}$

$C(x) = \frac{1}{2} x^{-1/2}$

$K(x) = \frac{1}{2}$

This is well conditioned in relative sense. It is also well conditioned in absolute sense except when x approaches 0

Example. $f(x) = \sin(x)$

$f'(x) = \cos(x)$

$C(x) = |\cos(x)| \leq 1$

$K(x) = |x \cot(x)|$

$K(x)$ is large near $\pm\pi, \pm 2\pi$ etc and when $|x|$ is large and $|\cot x|$ is not very small.

Stability of Algorithms.

A stable algorithm achieves the level of accuracy defined by the conditioning of the problem.

We can do a rounding error analysis to assess the stability of an algorithm.

Example : Computing Sums

$$\text{float}(x+y) \equiv \text{float}(x) \oplus \text{float}(y)$$
$$= (x(1+\delta_1) + y(1+\delta_2))(1+\delta_3) \qquad |\delta_i| < \epsilon$$

Forward error analysis.

How much does the computed value differ from exact value :

$$\text{float}(x+y) = x+y + (\delta_1 + \delta_3 + \delta_1\delta_3)x + (\delta_2 + \delta_3 + \delta_2\delta_3)y$$

$$|\text{float}(x+y) - (x+y)| \leq (|x| + |y|)(2\epsilon + \epsilon^2)$$

Relative error :

$$\frac{|\text{float}(x+y) - (x+y)|}{|x+y|} \leq \frac{(|x| + |y|)(2\epsilon + \epsilon^2)}{|x+y|}$$

This is large when $y \approx -x$

Backward error analysis.

Show that the computed value is the solution to a nearby problem.

$$\text{float}(x+y) = x(1+\delta_1)(1+\delta_3) + y(1+\delta_2)(1+\delta_3)$$

This is the true sum of two numbers that are atmost $2\epsilon + \epsilon^2$ away from x & y

$\therefore$ the solution is backward stable.

Example: Compute exp(x) using a Taylor series expansion.

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Compute this in a loop. The loop body is essentially (initialize previous_term = 1, expx = 0)

```
{
  expx = expx + previous_term x x/n ;
  previous_term = previous_term x x/n ;
}
```

Consider $f(-20)$; $\quad C(x)\big|_{x=-20} = \frac{d}{dx} e^x \big|_{x=-20} \ll 1$

$$K(x)\big|_{x=20} = \left| x e^x / e^x \right|_{x=20} = 20$$

This is well conditioned, but your code does not compute accurately.

How does one fix this code for negative x ?