Conditioning of linear Systems.

$$Ax = b$$

Input is (A, b)

Output is x.

We can talk of conditioning of this problem.

Norms.

A norm for vectors satisfies:

(i) $\|v\| \geq 0$, $\|v\| = 0$ iff $v = 0$

(ii) $\|\alpha v\| = |\alpha| \|v\|$ for any scalar α

(iii) $\|v + w\| \leq \|v\| + \|w\|$ (triangle inequality)

eg. 2- norm.

$$\|v\|_2 \equiv \sqrt{\sum_{i=1}^{n} |v_i|^2}$$

∞ norm

$$\|v\|_\infty \equiv \max_{i=1,n} |v_i|$$

More generally

$$\|v\|_p \equiv \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}$$

1 norm $\|v\|_1 \equiv \sum_{i=1}^{n} |v_i|$

Matrix Norm.

Must satisfy the following:

(i) $\|A\| \geq 0$, $\|A\| = 0$ iff $A = 0$

(ii) $\|\alpha A\| = |\alpha| \, \|A\|$ for any scalar α

(iii) $\|A+B\| \leq \|A\| + \|B\|$ (triangle inequality)

Some definitions also require

(iv) $\|AC\| \leq \|A\| \cdot \|C\|$ (Submultiplicative).

If $\|\cdot\|$ is a vector norm, the induced matrix norm is

$$\|A\| \equiv \max_{\|v\|=1} \|Av\| = \max_{v \neq 0} \frac{\|Av\|}{\|v\|}$$

$$\|A\| \geq \frac{\|Av\|}{\|v\|} \quad \text{or} \quad \|Av\| \leq \|A\| \cdot \|v\| \qquad (1)$$

Any vector norm that satisfies (1) is said to be compatible with or subordinate to matrix norm $\|\cdot\|$.

Thm.

A : $m \times n$ matrix

$\| A \|_1$ is the matrix norm induced by the 1-norm of vectors.

$\| A \|_1$ is the maximum absolute column sum

$$\| A \|_1 = \max_{j=1..n} \sum_{i=1}^{m} |a_{ij}|$$

Proof:

$$A = (a_{:,1}, \dots a_{:,n})$$

$$a_{:,j} = (a_{1j}, \dots a_{mj})^T$$

(jth column of A).

$$\| Av \|_1 = \| \sum_{j=1}^{n} a_{:,j} v_j \|_1 \le \sum_{j=1}^{n} |v_j| \cdot \| a_{:,j} \|_1$$

(triangle inequality).

$$\le \max_j \| a_{:,j} \|_1 \left(\sum_{j=1}^{n} |v_j| \right)$$

$$\le \max_j \| a_{:,j} \| \cdot \| v \|_1$$

If the index of the column with max 1-norm is J

and if $v_j = 1$ and all entries in v are 0 then

$$Av = a_{:,j}.$$

Also $\|v\|_1 = 1$ and

$$\|Av\|_1 = \max_j \|a_{:,j}\|.$$

Therefore $\|A\|_1 \geq \max_j \|a_{:,j}\|_1$

Thm: A: $m \times n$ matrix

$\|A\|_\infty$ is the norm induced by ∞-norm for vectors

Then $\|A\|_\infty$ is the max absolute row sum.

$$\|A\|_\infty = \max_{i=1..m} \sum_{j=1}^{n} |a_{ij}|$$

Proof:

$$\|Av\|_\infty = \max_{i=1..m} |(Av)_i| = \max_{i=1..m} \left| \sum_{j=1} a_{ij} v_j \right|$$

$$\leq \max_{i=1..m} \sum_{j=1}^{n} |a_{ij}| |v_j|$$

$$\leq \max_{i=1..m,\ j=1..n} (\max |v_j|) \sum_{j=1}^{n} |a_{ij}| = \|v\|_\infty \max_{i=1..m} \sum_{j=1}^{n} |a_{ij}|$$

If I is the index of a row with max absolute row sum and if $|v_j| = 1$ and

$a_{Ij} v_j = |a_{Ij} v_j| = |a_{Ij}|$ for all j (i.e., $v_j = 1$ if $a_{Ij} \geq 0$ and $v_j = -1$ if $a_{Ij} < 0$) then we will have equality above.

Thm. A: $m \times n$ matrix

$\|A\|_2$ is induced by vector 2-norm

$\|A\|_2$ is the square root of largest eigenvalue of $A^T A$.

Sensitivity of solving linear Systems.

Hilbert Matrix $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & & & & \vdots \\ \vdots & & & & \vdots \\ \frac{1}{n} & & & & \frac{1}{2n-1} \end{bmatrix}$ is

notoriously sensitive. Why?

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Say we want to solve

$$Ax = b \quad — (1)$$

We perturb b to $\hat{b}$ and we solve the system to get $\hat{x}$

$$A\hat{x} = \hat{b} \quad — (2)$$

From (1) and (2), we get

$$A(x - \hat{x}) = b - \hat{b}$$

or $(x - \hat{x}) = A^{-1}(b - \hat{b})$

or $\| x - \hat{x} \| \leq \|A^{-1}\| \, \| b - \hat{b} \|$

$\|A^{-1}\|$ can be thought of as the absolute condition number.

Relative error:

$$\frac{\|x-\hat{x}\|}{\|x\|} \leq \|A^{-1}\| \frac{\|b-\hat{b}\|}{\|x\|}$$

$$\leq \|A^{-1}\| \frac{\|b-\hat{b}\|}{\|b\|} \cdot \frac{\|b\|}{\|x\|}$$

now $\frac{\|b\|}{\|x\|} = \frac{\|Ax\|}{\|x\|} \leq \|A\|$

$$\therefore \frac{\|x-\hat{x}\|}{\|x\|} \leq \|A^{-1}\| . \|A\| . \frac{\|b-\hat{b}\|}{\|b\|}$$

We can think of $\|A^{-1}\| \|A\|$ as the relative condition number. This is also called $\kappa(A)$ ___ . or Condition number of A

Example. $A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$ $A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$

In 1-norm, $\kappa_1(A) = \|A\|_1 . \|A^{-1}\|_1$

$= 3 \cdot 1 = 3$

In ∞ norm, $\kappa_\infty(A) = 4 \cdot \frac{3}{4} = 3$.

In 2. norm.

$$A^TA = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad A^{-T}A^{-1} = \frac{1}{16}\begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$$

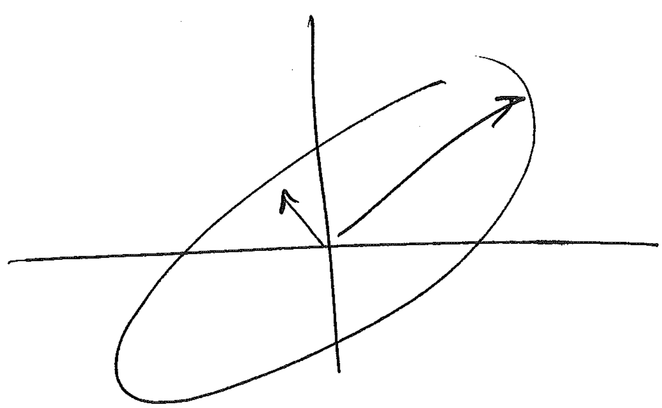
largest eigenvalue of A^TA is 8

$\therefore \quad \|A\|_2 = \sqrt{8} = 2\sqrt{2}$.

largest eigenvalue of $A^{-T}A^{-1}$ is $\frac{1}{2}$

$\therefore \quad \|A^{-1}\| = \frac{1}{\sqrt{2}}$

$\therefore$ Condition number is $2\sqrt{2} \times \frac{1}{\sqrt{2}} = 2$.



Ratio of the major axis to minor axis is the condition number.

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Suppose $\hat{b} = b + (\epsilon_1, \epsilon_2)^T$

$$\hat{x} = x + \underbrace{\frac{1}{4}\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}}_{A^{-1}}\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

$$= x + \frac{1}{4}\begin{pmatrix} 2\epsilon_1 + \epsilon_2 \\ -2\epsilon_1 + \epsilon_2 \end{pmatrix}.$$

Say $\epsilon_1 = \epsilon_2 = \epsilon > 0$

$$x - \hat{x} = -\frac{1}{4}\begin{pmatrix} 3\epsilon \\ -\epsilon \end{pmatrix}$$

$$\|x - \hat{x}\|_\infty = \underbrace{\frac{3}{4}}_{\uparrow}\epsilon$$

$\|A^{-1}\|_\infty$ (Absolute condition number)

Say $b = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \quad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\|x\|_\infty = 1 \qquad \|b\|_\infty = 4.$$

$\therefore$ with $\epsilon_1 = \epsilon_2 \equiv \epsilon$

$$\frac{\|x - \hat{x}\|_\infty}{\|x\|_\infty} = \frac{3}{4}\epsilon = 3\,\frac{\|b - \hat{b}\|_\infty}{\|b\|_\infty}$$

here $\kappa(A) = 3$ is the relative cond. number in infinite norm.

Thm:

A: non singular $n \times n$ matrix

b: n-vector.

x : $Ax = b$

$A+E$ is another nonsingular matrix

$\hat{b}$ is another n vector

$\hat{x}$ satisfies

$(A+E)\hat{x} = \hat{b}$

Then

$$\frac{\| x - \hat{x} \|}{\| x \|} \leq \left(\| (A+E)^{-1} \| \cdot \| A \| \right) \left(\frac{\| b - \hat{b} \|}{\| b \|} + \frac{\| E \|}{\| A \|} \right).$$

If $\| E \|$ is small enough so that

$\| A^{-1} \| \| E \| < 1$ then

$$\frac{\| x - \hat{x} \|}{\| x \|} \leq \frac{\kappa(A)}{1 - \kappa(A) \| E \| / \| A \|} \cdot \left(\frac{\| b - \hat{b} \|}{\| b \|} + \frac{\| E \|}{\| A \|} \right)$$