Polynomial and Piecewise Polynomial Interpolation.

Given $(n+1)$ data points

$(x_0\ y_0)\quad(x_1\ y_1)\ \cdots\quad(x_n\ y_n)$

find a polynomial of degree atmost n that exactly fits these points.

$$y_i = \sum_{j=0}^{n} c_j x_i^j \qquad i = 0, 1, \ldots n$$

plugging in each of the $(n+1)$ points, we have

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & & \vdots & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

This is called the vandermonde form.

The matrix is typically ill conditioned and the system is expensive to solve.

Can we do better?

Lagrange form.

Once again, given points

$(x_0, y_0), (x_1, y_1) \dots (x_n, y_n)$

what is a polynomial that is nonzero only at x_0 but zero at $x_1, x_2 \dots x_n$?

$$(x-x_1)(x-x_2)(x-x_3)\dots(x-x_n)$$

what is the polynomial that is 1 at x_0 and zero everywhere else?

$$\cancel{(x_1 - x_0} \quad \frac{(x-x_1)(x-x_2)(x-x_3)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}$$

what is the polynomial that is y_0 at x_0 and zero at all other points?

$$\frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} \cdot y_0$$

Do the same for all the other points and add the polynomials to get the answer!

$$\phi_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

$$p(x) = \sum_{i=0}^{n} y_i \, \phi_i(x)$$

This is called the lagrange form.

Easy to write, but it takes n^2 time to evaluate.

It has $O(n)$ terms, each requiring $O(n)$ operations.

Example. Find a degree 2 polynomial that goes through

$(1,2)$, $(2,3)$, $(3,6)$

$$\phi_0(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} \qquad \phi_1(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)}$$

$$\phi_2(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)}$$

$$P(x) = 2 \cdot \phi_0(x) + 3 \cdot \phi_1(x) + 6 \cdot \phi_2(x)$$

This is $x^2 - 2x + 3$.

Newton form.

Consider the form

$$p(x) = a_0 + a_1 (x-x_0) + a_2 (x-x_0)(x-x_1) + \dots + a_n (x-x_0)(x-x_1)\dots(x-x_n)$$

What happens when $x = x_0$?

$$p(x_0) = a_0 = f(x_0) = y_0$$

$$p(x_1) = a_0 + a_1 (x_1 - x_0) = y_1$$

$$p(x_2) = a_0 + a_1 (x_2 - x_0) + a_2 (x_2-x_0)(x_2-x_1) = y_2$$

$$\vdots$$

$$A = \begin{bmatrix} 1 & & & & \\ 1 & (x_1 - x_0) & & & \\ 1 & (x_2 - x_0) & (x_2-x_0)(x_2-x_1) & & \\ \vdots & & & & \\ 1 & (x_n - x_0) & (x_n - x_0)(x_n - x_1) & \dots & (x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1}) \end{bmatrix}$$

$$A \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Example.

Consider again $(1,2),(2,3),(3,6)$

In newton form, we get

$$\begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}.$$

$a_0 = 2, \quad a_1 = 1, \quad a_2 = 1$

we get $p(x) = x^2 - 2x + 3$ again.

Example.

Say we now add another point $(5,7)$

The system becomes

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 2 & \\ 1 & 4 & 12 & 24 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \\ 7 \end{pmatrix}$$

The values of a_0, a_1, a_2 do not change

we can immediately substitute and find

$$a_3 = -\frac{11}{24}$$

Divided difference form.

Say you are given two points

$$(x_0, y_0), \ (x_1, y_1)$$

and you want to fit a line using newtons form

$$p(x) = a_0 + a_1 (x - x_0).$$

At point $x = x_0$, $p(x_0) = f(x_0)$

$$\therefore \ f(x_0) = a_0$$

At point $x = x_1$, $p(x_1) = f(x_1)$

$$f(x_1) = f(x_0) + a_1 (x_1 - x_0)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = F[x_1, x_0]$$

$$p(x) = f(x_0) + F[x_1, x_0] \cdot (x - x_0)$$

Now consider a quadratic

$(x_0\ y_0)\quad (x_1\ y_1)\quad (x_2\ y_2)$

$$p(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \cdot (x - x_0)$$

$$+ a_2\,(x - x_0)(x - x_1)$$

We just added (x_2, y_2) and we had noted that this does not change prev. coefficients.

At (x_2, y_2) we have

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} \cdot (x_2 - x_0)$$

$$+ a_2\,(x_2 - x_0)(x_2 - x_1)$$

Solving for a_2, we get

$$a_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{F[x_2, x_1] - F[x_1, x_0]}{x_2 - x_0} = F[x_2, x_1, x_0]$$

In general,

$$a_0 = f(x_0)$$

$$a_1 = F[x_1, x_0]$$

$$a_2 = F[x_2\, x_1\, x_0]$$

$$\vdots$$

$$a_m = F[x_m \dots x_0] \qquad 1 \le m \le n$$

$$F[x_m \dots x_0] = \frac{F[x_m \dots x_1] - F[x_{m-1} \dots x_0]}{x_m - x_0}$$

Computing these divided differences.

$$\begin{array}{cccccc}
 & \overset{a_0}{\swarrow} & & \overset{a_1}{\swarrow} & \overset{a_2}{\swarrow} & \overset{a_3}{\swarrow} \\
x_0 & y_0 & & & & \\
 & & F[x_1, x_0] & & & \\
x_1 & y_1 & & F[x_2, x_1, x_0] & & \\
 & & F[x_2, x_1] & & F[x_3 \dots x_0] & \\
\vdots & \vdots & & F[x_3, x_2, x_1] & & \\
x_n & y_n & & & &
\end{array}$$

Divided Difference Example

$(1, 2) \quad (2, 3) \quad (3, 6)$

X	Y	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+2}]$	
		a_0		
1	2			
			a_1	
2	3	1		
				a_2
3	6	3	1	

$$\therefore \quad p(x) = 2 + 1(x-1) + 1(x-1)(x-2)$$