Error in Polynomial Interpolation

<u>Thm</u>:

Assume $f \in C^{n+1}[a,b]$,

$x_0 \dots x_n \in [a,b]$

$p(x)$ interpolates f at $x_0 \dots x_n$

Then for any $x \in [a,b]$

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{j=0}^{n} (x - x_j)$$

for some point ξ_x in $[a,b]$

<u>Proof</u>. Add another point x to $x_0 \dots x_n$ $(x \neq x_i)$.

let q be the polynomial that interpolates f at

$x_0, x_1 \dots x_n, x$

$$q(t) = p(t) + \lambda \prod_{j=0}^{n} (t - x_i), \quad \lambda = \frac{f(x) - p(x)}{\prod_{j=0}^{n} (x - x_i)} \quad \text{— (1)}$$

(from Lagrange form).

Define $\phi(t) \equiv f(t) - q(t)$

$\phi(t)$ vanishes at $n+2$ points $(x_0 \ldots x_n, x)$.

$\phi'(t)$ vanishes at $n+1$ points.

$\phi''(t)$ vanishes at n points.

$\phi^{n+1}(t)$ vanishes at atleast one point in $[a,b]$

Call this point ξ_x

$$\phi^{n+1}(\xi_x) = 0 = f^{(n+1)}(\xi_x) - q^{(n+1)}(\xi_x) \quad \text{——} (2)$$

Differentiating (1) (page (1)) $n+1$ times, we get

$$q^{(n+1)}(t) = p^{(n+1)}(t) + \lambda(n+1)!$$

$p^{(n+1)}(t) = 0$ Since it is a degree n polynomial

$$\therefore q^{(n+1)}(t) = \lambda(n+1)!$$

from (2) we get

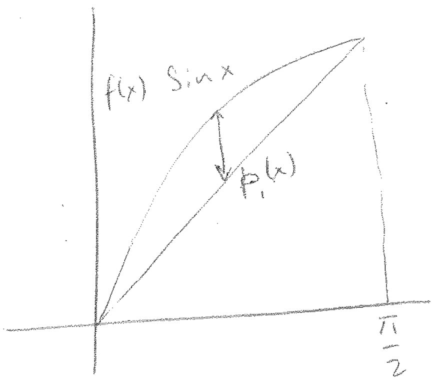
$$f^{(n+1)}(\xi_x) = \lambda(n+1)! = \frac{f(x) - p(x)}{\prod_{j=0}^{n}(x - x_j)} \cdot (n+1)!$$

□

Example.

$f(x) = \sin x$

p_1 : degree 1 polynomial that interpolates f at 0 and $\frac{\pi}{2}$



$$|f(x) - p_1(x)| \le \frac{1}{2!} \left|(x-0)\left(x-\frac{\pi}{2}\right)\right|$$

This has max value at $x = \frac{\pi}{4}$

$$|f(x) - p_1(x)| \le \frac{1}{2}\left(\frac{\pi}{4}\right)^2$$

The actual error is

$$\left|\sin\left(\frac{\pi}{4}\right) - \frac{2}{\pi}\cdot\frac{\pi}{4}\right| = \frac{(\sqrt{2}-1)}{2} \approx 0.207$$

Can we extrapolate to $f(\pi)$?

Chebyshev Interpolation

We learned that error in interpolation is

$$f(x) - p_n(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_n)}{(n+1)!} \cdot f^{(n+1)}(\xi_x)$$

The max. error is

$$\leq \max \frac{|(x-x_0)(x-x_1)\cdots(x-x_n)|}{(n+1)!} \max |f^{(n+1)}(\xi_x)|$$

Example

Consider 11 equispaced points on $[-1, 1]$

$x = -1, -0.8, -0.6, \cdots, 0, 0.2, 0.4, \cdots 0.8, 1$

Consider the $|(x-x_0)\cdots(x-x_n)|$ term in the error

Near $x = 0.1$ this term is

$1.1 \times 0.9 \times 0.7 \times \cdots \times -0.9 \approx -0.0000982$

Near $x = 0.9$ this term is

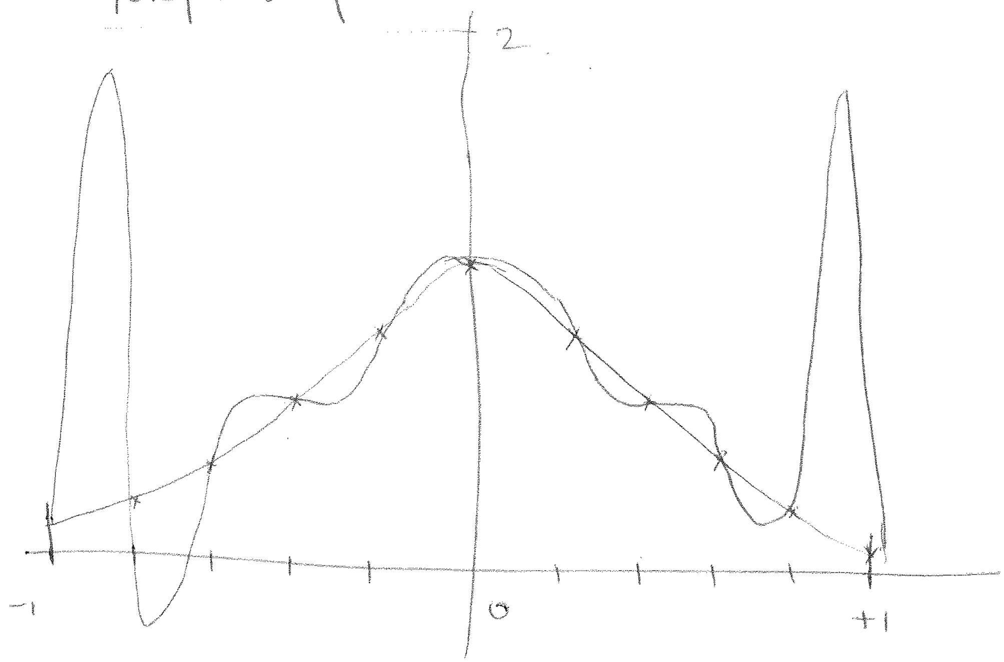
$1.9 \times 1.7 \times 1.5 \cdots \times -0.1 \approx -0.00655$

Error is much larger around the ends than in the center!

Example

$$f(x) = \frac{1}{1+25x^2}$$

11 equispaced points



Max error ~ 1.92

Pick points smarter -- more around the corners than in the middle

These are the roots of the Chebyshev polynomials

Error using Chebyshev points ~ 0.11

Improvement of a factor of 17!

The special points:

In order to minimize the max value of the numerator, given that we have $n+1$ points, we use the following points.

$$x_j = \cos\left(\frac{\pi j}{n}\right) \quad j=0,1,\dots n.$$

These are the roots of the Chebyshev polynomial.

One can show that the max value of the numerator is

$$\max |(x-x_0)(x-x_1)\dots(x-x_n)| = \frac{1}{2^n}$$

Example

Consider $f(x) = \frac{1}{1+x^2}$ on $[-5, 5]$

Change variable from x to t

$t = \frac{x}{5}$ $\quad t \in [-1, 1]$

$g(t) = \frac{1}{1+25t^2}$.

$t_j = \cos\left(\frac{\pi j}{n}\right)$ $\quad j = 0, 1, \dots n$

n	$\lVert g - p_n \rVert_\infty$	$\lVert g - p_n \rVert_\infty / \lVert g - p_{n/2} \rVert_\infty$
5	0.64	
10	0.13	0.21
20	0.018	0.13
40	3.33×10^{-4}	0.019

explained by $\lVert f - p_n \rVert_\infty = O(C^n)$

$C \approx 0.82$.

So how good are these "Special Points"?

Thm let f be continuous on $[-1, 1]$,

P_n is the degree n interpolant on Chebychev points.

P_n^* is the best approximation among all n-degree polynomials on $[-1, 1]$ in the ∞ norm. Then:

1. $\| f - P_n \|_\infty \le \left(2 + \frac{2}{\pi} \log n \right) \| f - P_n^* \|_\infty$

2. If f has k^{th} derivative of bounded variation (total vertical distance along the graph of the function is finite) in $[-1, 1]$ for some $k \geq 1$ then

$$\| f - P_n \|_\infty = O(n^{-k}) \text{ as } n \to \infty$$

3. If f is analytic in a neighborhood of $[-1, 1]$ then $\| f - P_n \|_\infty = O(c^n)$ as $n \to \infty$ and $c < 1$.

Bounded variation

f is defined on $[a,b]$ and

$P: \{a = x_0, x_1 \dots x_n = b\}$ is a partition.

Consider the sum $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$

If this set of sums is bounded above then the function f is said to be a bounded variation.

Real Analytic Functions

Say we can compute

$f'(0)\ f''(0) \dots$ infinitely differentiable at zero.

Then $f(x) \overset{?}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

f is real analytic at 0 if $\exists\, R > 0 \ni$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{when } |x| < R$$