# Piecewise Polynomial Interpolation.

Instead of fitting a single polynomial over $[a,b]$, divide $[a,b]$ into smaller intervals and fit a (lower degree) polynomial on these sub-intervals.

$[a,b]$ is divided into $x_0, x_1, \ldots, x_n$

$$x_0 = a, \quad x_n = b.$$

$l(x)$ is the piecewise linear interpolant.

then on subinterval $[x_{i-1}, x_i]$

$$l(x) = f(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + f(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}}$$

Why might you do this?

Ex Interpolating complex function.

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left[ f_{i-1} \frac{x - x_i}{x_{i-1} - x_i} + f_i \frac{x - x_{i-1}}{x_i - x_{i-1}} \right] dx$$

$$= \sum_{i=1}^{n} \left[ f_{i-1} \cdot \frac{h}{2} + f_i \frac{h}{2} \right] \qquad \text{here, } x_i - x_{i-1} = h$$

$$= \frac{h}{2} \left[ f_0 + 2f_1 + \cdots + 2f_{n-1} + f_n \right]$$

This is the trapezoid rule for integration.

Ex Interpolation in a table.

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

What is $\sin\left(\frac{\pi}{5}\right)$?

$$l(x) = \frac{1}{2} \cdot \frac{x - \frac{\pi}{4}}{\frac{\pi}{6} - \frac{\pi}{4}} + \frac{\sqrt{2}}{2} \cdot \frac{x - \frac{\pi}{6}}{\frac{\pi}{4} - \frac{\pi}{6}}$$

$$l\left(\frac{\pi}{5}\right) \approx 0.58$$

Error in linear Interpolation.

$$f(x) - l(x) = \frac{f''(\xi)}{2!}(x - x_{i-1})(x - x_i)$$

Say $f''(\xi) \le M$ in $[x_{i-1}, x_i]$

Then

$$|f(x) - l(x)| \le \frac{M}{2} \max |(x - x_{i-1})(x - x_i)|$$

$$\le \frac{M}{2} \cdot \frac{h^2}{2} = \frac{M}{8} h^2$$

If we assume that $f''(\xi) \le M$ over $[a, b]$,

Over the entire interval, the bound holds.

$$|f(x) - l(x)| \le \frac{Mh^2}{8} \text{ for all } x \in [a, b]$$

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We can alter (reduce $h$) to match desired error $\delta$

$$h < \sqrt{8\delta/M}$$

Also the piecewise interpolant converges to the function as $h \to 0$.

Reducing error.

We do not have to sample uniformly.

Divide intervals where you see max errors.

Compare $f\left(\frac{x_i + x_{i-1}}{2}\right)$ and $l\left(\frac{x_i + x_{i-1}}{2}\right)$

If this error exceeds a threshold, then

divide the interval.

This is only a heuristic!

We can also reduce error by increasing the degree of the polynomial. This is called p-refinement as opposed to h-refinement earlier.

# Piecewise Cubic Hermite Interpolation.

Match both function values <u>and</u> derivatives at the boundary points of the interval.

$$f(x_{i-1}) = p(x_{i-1}) \qquad f(x_i) = p(x_i)$$

$$f'(x_{i-1}) = p'(x_{i-1}) \qquad f'(x_i) = p'(x_i)$$

You are looking to fit a cubic polynomial.

$\therefore$ 4 unknowns and 4 equations.

You can simply write

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

and solve for $c_0, c_1, c_2, c_3$.

This is not convenient or efficient.

Piecewise Cubic Hermite Interpolation.

$$p'(x) = f'(x_{i-1}) \cdot \frac{x - x_i}{x_{i-1} - x_i} + f'(x_i)\frac{x - x_{i-1}}{x_i - x_{i-1}}$$

$$+ \alpha\,(x - x_{i-1})(x - x_i) \quad \text{(1)}$$

$x \in [x_{i-1}, x_i]$

use this to match function values ~~[illegible]~~

Integrating (1) we get

$$p(x) = -\frac{f'(x_{i-1})}{h}\int_{x_{i-1}}^{x}(t - x_i)\,dt + \frac{f'(x_i)}{h}\int_{x_{i-1}}^{x}(t - x_{i-1})\,dt$$

$$+ \alpha\int_{x_{i-1}}^{x}(t - x_{i-1})(t - x_i)\,dt + C$$

we know that $p(x_{i-1}) = f(x_{i-1})$.

Substituting above, we get

$C = f(x_{i-1})$

$$\therefore \quad p(x) = -\frac{f'(x_{i-1})}{h}\left(\frac{(x-x_i)^2}{2} - \frac{h^2}{2}\right)$$

$$+ \frac{f'(x_i)}{h}\frac{(x-x_i)^2}{2}$$

$$+ \alpha (x-x_i)^2 \left(\frac{x-x_{i-1}}{2} - \frac{h}{2}\right) + f(x_{i-1})$$

We also know that $p(x_i) = f(x_i)$

$$\therefore \quad p(x_i) = f'(x_{i-1}) \cdot \frac{h}{2} + f'(x_i) \cdot \frac{h}{2} - \alpha\frac{h^3}{6} + f(x_{i-1}) = f(x_i)$$

$$\therefore \quad \alpha = \frac{2}{h^2}\left(f'(x_{i-1}) + f'(x_i)\right) + \frac{6}{h^3}\left(f(x_{i-1}) - f(x_i)\right).$$

$\square$

Example.

$f(x) = x^4$ on $[0,2]$

find the piecewise cubic Hermit interpolant for $f$ on $[0,1]$ and $[1,2]$

$x_0 = 0 \quad x_1 = 1 \quad x_2 = 2, \quad h = 1$

Solving the equations above, we get

$$p(x) = \begin{cases} 2x^3 - x^2 & \text{on } [1,2] \\ 6x^3 - 13x^2 + 12x - 4 & \text{on } [1,2] \end{cases}$$