Cubic Spline Interpolation.

Given $f(x)$ in interval $[a, b]$

define nodes (a.k.a. knots) $x_0, x_1, \ldots x_{n+1}$

$x_0 = a, \quad x_{n+1} = b$

We have n intervals of the form $[x_{i-1}, x_i]$

$i = 1 \ldots n$.

Define an interpolant S_i over this interval, so that

$$\left.\begin{array}{ll} ① & S_i(x_{i-1}) = f(x_{i-1}) \\ ② & S_i(x_i) = f(x_i) \end{array}\right\} \text{function value matches at knots.}$$

$$\left.\begin{array}{ll} ③ & S_i'(x_{i-1}) = S_{i-1}'(x_{i-1}) \\ ④ & S_i'(x_i) = S_{i+1}'(x_i) \end{array}\right\} \text{Derivatives of the interpolants match at nodes.}$$

$$\left.\begin{array}{ll} ⑤ & S_i''(x_{i-1}) = S_{i-1}''(x_{i-1}) \\ ⑥ & S_i''(x_i) = S_{i+1}''(x_i) \end{array}\right\} \text{Second derivatives match.}$$

S_i has 4 parameters (cubic polynomial)

There are n intervals (S_i's)

$\therefore$ we have $4n$ unknowns.

The function must match the interpolant at all nodes $(x_0, x_1 \ldots x_n)$ .. $n+1$ nodes.

- For each of the internal nodes, we get two equation. (one from left, one from right)
- For the two external nodes, we get two equations (total)

$\therefore$ we have $2(n-1)+2 = 2n$ equations.

- At each internal node, we also get two equations (matching the derivative and second derivative)

$\therefore$ we have $2(n-1)$ equations.

$\therefore$ In all we have $2n + 2(n-1) = 4n-2$ equations and $4n$ unknowns.

We need two more equations.

Natural splines.

The second derivative at external nodes is 0

Complete spline.

First derivative of the interpolant matches the first derivative of the function at external nodes.

These give 2 more equations.

$\therefore$ $4n$ variables, $4n$ equations.

Solve the linear system!

Deriving equations for Splines.

Say the second derivatives at nodes are

$z_1, z_2 \ldots z_{n-1}$

Assume for now that we know these.

Since we have a cubic spline, S'' is linear.

$$\therefore \; S''(x) = z_{i-1} \frac{x - x_i}{x_{i-1} - x_i} + z_i \frac{x - x_{i-1}}{x_i - x_{i-1}} \quad \text{in } [x_{i-1}, x_i]$$

$$= -\frac{1}{h} z_{i-1} (x - x_i) + \frac{1}{h} z_i (x - x_{i-1})$$

Integrating this, we get

$$S_i'(x) = -\frac{1}{h} z_{i-1} \frac{(x - x_i)^2}{2} + \frac{1}{h} z_i \frac{(x - x_i)^2}{2} + c_i \qquad (1)$$

$$S_i(x) = -\frac{1}{h} z_{i-1} \frac{(x - x_i)^3}{6} + \frac{1}{h} z_i \frac{(x - x_{i-1})^3}{6}$$

$$+ C_i (x - x_{i-1}) + D_i$$

we know that $S_i(x_{i-1}) = f_{i-1}$

$$\therefore \quad \frac{h^2}{6} z_{i-1} + D_i = f_{i-1} \quad \text{or} \quad D_i = f_{i-1} - \frac{h^2}{6} z_{i-1}$$

$S_i(x_i) = f_i$

$$\therefore \quad \frac{h^2}{6} z_i + C_i h + f_{i-1} - \frac{h^2}{6} z_{i-1} = f_i$$

$$\text{or} \quad C_i = \frac{1}{h}\left[f_i - f_{i-1} + \frac{h^2}{6}(z_{i-1} - z_i) \right]$$

$$\therefore \quad S_i(x) = -\frac{1}{h} z_{i-1} \frac{(x - x_i)^3}{6} + \frac{1}{h} z_i \frac{(x - x_{i-1})^3}{6}$$
$$+ \frac{1}{h}\left[f_i - f_{i-1} + \frac{h^2}{6}(z_{i-1} - z_i) \right](x - x_{i-1})$$
$$+ f_{i-1} - \frac{h^2}{6} z_{i-1}.$$

How do we find z_i s ?

Use continuity of first derivatives.

$$S_i'(x_i) = S_{i+1}'(x_i)$$

Plug this into 1 on page 4.

$$\frac{h}{2} z_i + \frac{1}{h}(f_i - f_{i-1}) + \frac{h}{6}(z_{i-1} - z_i) = -\frac{h}{2} z_i + \frac{1}{h}(f_{i+1} - f_i) + \frac{h}{6}(z_i - z_{i+1})$$

Move all unknown zs to LHS and known f_is to RHS.

$$\frac{2h}{3} z_i + \frac{h}{6} z_{i-1} + \frac{h}{6} z_{i+1} = -\frac{2}{h} f_i + \frac{1}{h} f_{i-1} + \frac{1}{h} f_{i+1}$$

$$i = 1, 2, \ldots, n-1$$

$$\begin{bmatrix} \alpha_1 & \beta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \beta_{n-2} \\ & & \beta_{n-2} & \alpha_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_{n-1} \end{bmatrix}$$

Here

$$\alpha_i = \frac{2h}{3} \qquad \beta_i = \frac{h}{6}$$

$$b_i = \frac{1}{h}\left(f_{i+1} - 2f_i + f_{i-1}\right) \qquad i = 2, \ldots n-2$$

$$b_1 = \frac{1}{h}\left(f_2 - 2f_1 + f_0\right) - \frac{h}{6} z_0$$

$$b_{n-1} = \frac{1}{h}\left(f_n - 2f_{n-1} + f_{n-2}\right) - \frac{h}{6} z_n.$$

Example. $f(x) = x^4$ on $[0,2]$

Natural spline on $[0,1]$ and $[1,2]$

$n = 2$

The tridiagonal system consists of only one unknown (z_1).

$$\alpha_1 = \frac{2}{3} \qquad b_1 = f_2 - 2f_1 + f_0 = 14$$

$$\therefore \quad z_1 = 21, \qquad z_0 = z_2 = 0.$$

$$S_1(x) = \frac{7}{2}x^3 - \frac{5}{2}x$$

$$S_2(x) = -\frac{7}{2}x^3 + 21x^2 - \frac{47}{2}x + 7.$$