Richardson Extrapolation.

$$f'(x) = \frac{4}{3} \delta_0\left(\frac{h}{2}\right) - \frac{1}{3} \delta_0(h) + O(h^4)$$

Twice as much work as centered difference since we evaluate δ_0 at h and $h/2$.

Truncation error is $O(h^4)$.

Rounding error is still $\frac{\epsilon}{h}$.

$$\therefore \quad h^4 \sim \frac{\epsilon}{h} \quad \text{or} \quad h \simeq \epsilon^{1/5}$$

$$\text{Error} = \epsilon^{4/5}$$ (almost to machine precision).

We can recursively repeat this process.

$$f'(x) = \delta_1(h) + Ch^4 + O(h^5). \quad \text{—— (1)}$$

(we have written the $O(h^4)$ term as $Ch^4 + O(h^5)$)

With spacing $\frac{h}{2}$, we have

$$f'(x) = \delta_1\left(\frac{h}{2}\right) + C\left(\frac{h}{2}\right)^4 + O(h^5). \quad \text{— (2)}$$

$2^4 \times$ (2) $-$ (1) gives us

$$f'(x) = \frac{16}{15}\, \phi_1\!\left(\frac{h}{2}\right) - \frac{1}{15}\, \phi_1(h) + O(h^6)$$

(the h^4 and h^5 terms cancel)

$$\therefore \quad f'(x) = \frac{16}{15}\, \phi_1(h/2) - \frac{1}{15}\, \phi_1(h) + O(h^6).$$

Example. $f(x) = \sin x$

Compute $f'\left(\frac{\pi}{3}\right)$

$\phi_0(h)$: Centered difference formula.

$$f'\left(\frac{\pi}{3}\right) \approx \phi_0(h) = \frac{\sin\left(\frac{\pi}{3}+h\right) - \sin\left(\frac{\pi}{3}-h\right)}{2h}.$$

With $h = 10^{-3}$, error $= 5.5e^{-14}$

$h = 10^{-1}$, error $= 1.0e^{-7}$

Richardson.

$$\phi_2(h) = \frac{16}{15}\, \phi_1\!\left(\frac{h}{2}\right) - \frac{1}{15}\, \phi_1(h)$$

$h = 1e^{-1}$, error $= 1.6e^{-12}$

General use of Richardson Scheme.

Say
$$L = \underbrace{\phi_0(h)}_{\substack{\uparrow \\ \text{approximation}}} + \underbrace{a_1 h + a_2 h^2 + a_3 h^3 + \cdots}_{\text{error.}} \quad \text{— (1)}$$

Use the same approximation with step $\frac{h}{2}$.

$$L = \phi_0\left(\frac{h}{2}\right) + \frac{a_1}{2} h + \frac{a_2}{4} h^2 + \frac{a_3}{8} h^3 + \cdots \quad \text{— (2)}$$

(2) $\times 2$ − (1) gives

$$L = \underbrace{2\phi_0(h/2) - \phi_0(h)}_{\phi_1(h)} - \frac{1}{2} a_2 h^2 - \frac{3}{4} a_3 h^3 + \cdots$$

$$L = \phi_1(h) + b_2 h^2 + b_3 h^3 + \cdots \quad \text{— (3)}$$

$$L_1 = \phi_1(h/2) + \frac{b_2}{4} h^2 + \frac{b_3}{8} h^3 + \cdots \quad \text{— (4)}$$

(4) $\times 4$ − (3)

$$L = \underbrace{\frac{4}{3}\phi_1\left(\frac{h}{2}\right) - \frac{1}{3}\phi_1(h)}_{\phi_2(h)} - \frac{1}{6} b_3 h^3 + \cdots$$

$$L = \phi_2(h) + c_3 h^3 + c_4 h^4 + \cdots$$

Repeat.

Numerical Integration.

$$\int_a^b f(x)\,dx.$$

Newton - Cotes formulae.

Replace $f(x)$ by $p(x)$ and integrate $p(x)$.

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} f(x_i) \int_a^b \left(\prod_{\substack{i=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j} \right) dx$$

Formulae derived using uniform spacing are called Newton-Cotes formulae.

$n=1$

$$P_1(x) = f(a)\,\frac{x-b}{a-b} + f(b)\,\frac{x-a}{b-a}$$

$$\int_a^b f(x)\,dx \approx \int_a^b P_1(x)\,dx = \frac{f(a)}{a-b}\int_a^b (x-b)\,dx + \frac{f(b)}{b-a}\int_a^b (x-a)\,dx$$

$$= \frac{f(a)}{a-b}\left.\frac{(x-b)^2}{2}\right|_a^b + \frac{f(b)}{b-a}\left.\frac{(x-a)^2}{2}\right|_a^b$$

$$= \frac{b-a}{2}\left(f(a)+f(b)\right) \qquad \text{(trapezoid rule).}$$

Error in Newton Cotes:

Recall:

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{l=0}^{n} (x - x_l)$$

$$\therefore \int_a^b f(x)\,dx - \int_a^p p(x)\,dx$$

$$= \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\xi_x) \left(\prod_{l=0}^{n} (x - x_l) \right) dx$$

For $n = 1$ (trapezoid rule), this becomes

$$\frac{1}{2} \int_a^b f''(\xi_x)(x-a)(x-b)\,dx$$

$$= \frac{1}{2} f''(\eta) \int_a^b (x-a)(x-b)\,dx \qquad \eta \in [a,b]$$

$$= -\frac{1}{12}(b-a)^3 f''(\eta).$$