Lecture 16

⊛ No class or TA on Thursday ⊛

- HW 35 due Next Monday like usual.

- Why Condition # is

$$\frac{\|\underline{M}(\underline{x}+\underline{\delta}) - \underline{M}(\underline{x})\| \,/\, \|\underline{M}(\underline{x})\|}{\|\underline{\delta}\| / \|\underline{x}\|}$$

✓

- Estimate cond. of least sq. / linear system quickly?

- For one problem, adding a perturbation fixed it. ✓
  Why does this work?
- For pseudo-inverse why $\|AA^{+}\| = 1$ for op 2-norm ✓
- Inversion for $f(\underline{x}) = A^{-1}\underline{b}$ = A^{-1}. ✓

Way to show that some matrices cannot be solved by LU, QR?

Condition #s.

Idea: How sensitive a problem is:

⇒ Small Relative change to input creates a big Relative change to output.

for $M(x)$

$$\frac{\| M(x+\underline{\delta}) - M(\underline{x}) \|}{\|\underline{\delta}\|} \rightarrow \text{change in output}$$

change in input.

If $M(x) = 10^9$

$M(x+\varepsilon) = 10^{10}$

If $x = \varepsilon$, then
this is a big chang to x!

So

$$\|\delta\| \big/ \|x\| \rightarrow \text{Rel. change in } x$$

$$\frac{M(x+\delta) + M(x)}{M(x)} \rightarrow \text{Rel. chang in } M.$$

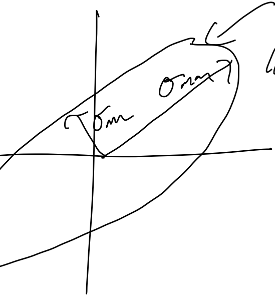
F

Fast Est. of

$$\kappa(A) = \|A\|_2 \cdot \|A^{-1}\|_2 = \boxed{\frac{\sigma_{max}}{\sigma_{min}}}$$

To be quick

Use Cond #
Estimators. or
a few steps of
Lanczos / Power
Method to
Estim $\sigma_{max}/\sigma_{min}$...



Ax for
$\|\underline{x}\| = 1$

$A\underline{x} = b$

$$A = \begin{bmatrix} 1 & & & & 1 \\ 1 & 1 & & & 1 \\ 1 & 1 & 1 & & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

In Julia, this doesn't work ☹.

Error = $2^n \varepsilon$.

Pseudo Inverse of a Matrix.

- A is ~~A~~ Sing. or Rect.

So A^{-1} doesn't exist.

- We can use a pseudo-inverse instead

$$A^+ = V \begin{bmatrix} 1/\sigma_1 & & & & & \\ & 1/\sigma_2 & & & & \\ & & \ddots & & & \\ & & & 1/\sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

$$A^+ A = V \Sigma^+ \overset{I}{\cancel{U^T U}} \Sigma V^T$$

u^T

$$\Sigma^+ \Sigma = \left[\begin{array}{cccc|ccc} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ \hline & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{array}\right]$$

$$A^+A = V\Sigma^+\Sigma V^T$$
$$= \sum_{i=1}^{r} v_i v_i^T$$

$$\|A^+A\|_2 = 1.$$

$$f(\underline{x}) = A^{-1}b$$
$$= M\underline{b}$$

$$f(x)_i = \sum_{j=1}^{n} M_{ij} b_j$$

$$\frac{\partial f_i}{\partial b_j} = M_{ij}$$

$$J(\underline{x}) = M.$$

$$J(\underline{x}) = M^T.$$

cience - David F. Gleich - 2024 CS 515 - Purdue University - Computer Science - David F. Gleich - 2024 CS 515 - Purdue University - Computer Science - David F. Gleich - 2024 CS

When do LU, QR fail?

- Never! They always work!

$x \oplus y = (x+y)($

x; Float64

xhi: Float64x4..

$(1+\varepsilon)$

$|\varepsilon| \le u = 2^{-53}$