# Lecture 24

- HW 10, 11
  - ↳ 10 due 20th
  - 11 "due" 27th
  - ↳ auto Extension to 12/2.

• Today CG via Lanczos

↳ Cholesky of tridiag matrices.

↳ Avoiding lots of Extra work!

→ We will start talking about GMRES.

Last time we talked about MINRES for $Ax = b$

[illegible], – Then we'll from Lanczos on $A, b \rightarrow V_k, T_{k+1}$

Recap of Lanczos / Krylov method,

Given $b$, we compute

$$V_k, \quad T_{k+1} = \begin{bmatrix} \overline{T_k} \\ \beta_{k+1} \underline{e}_k \end{bmatrix}$$

("strictly" $k+1 \times k$ Matrix.

$k \to k+1$, we get

One more vector

$\underline{V}_{k+1}$, & new values

$\alpha_{k+1}$, $\beta_{k+2}$

## CG vs MINRES

In both

$$x_k = \underline{V}_k y_k$$

MINRES picks $y_k$ s.t.

$\| b - Ax_k \|_2$ is

as small as possible.

In CG, we pick

$$y_k = \|b\| \overline{T}_k^{-1} \underline{e}_1$$

This gives

$$\| \, \|b\| e_1 - T_{k+1} y_k^{(CG)} \|_2 =$$

$$\left\| \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \hline \beta_1 e_k^T y_k^{(CG)} \end{bmatrix} \right\|_2$$

# ALG CG

$k \to k+1$

- Compute $V_k, T_k$ → $O(k$ matvecs$)$ → 1 matvec
  $O(n)$ work → $O(1)$ work.

- Compute/Solve $T_k y_k = \|b\| e_1$ → $O(k$ work$)$ → $O(1)$ to update factorization
  $O(k+1)$ to update soln

- Compute $x_k = V_k y_k$ → $O(nk$ work$)$
  → $O(nk+n)$ work

We will show how to get $x_{k+1}$ w/ $O(n + 1\ \text{matvec})$ work..

Step 1: Analyze the factorization of tridiag. matrices.

Work out Cholesky factorization of

$$\overline{T_k} = \begin{bmatrix} \alpha_1 & \beta_2 & & & \\ \beta_2 & \alpha_2 & \beta_3 & & \\ & & \ddots & & \\ & & & & \beta_k \\ & & & \beta_k & \alpha_k \end{bmatrix}$$

$$\overline{T_k} = F_k F_k^T.$$

Quiz What Cholesky fact.

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 5 \\ & -2 \end{bmatrix}$$

$$F = \begin{bmatrix} 2 & \\ -1 & 2 \\ & -1 \end{bmatrix}$$

is the

m of

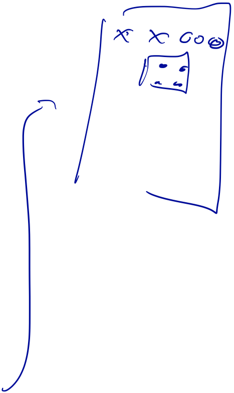
$$\begin{bmatrix} -2 & & \\ 5 & -2 & \\ -2 & 5 & \end{bmatrix}$$

$$\begin{bmatrix} 2 & \\ -x & 2 \end{bmatrix}$$

In general a Cholesky factorization of a tridiag matrix is bidiag.

$$\begin{bmatrix} x & x & & \\ x & x & x & \\ & x & x & x \\ & & x & x \end{bmatrix}$$

$\downarrow$ Stop!



Let's prove t

$$\begin{bmatrix} \alpha_1 & \beta_2 & & \\ \beta_2 & \alpha_2 & \beta_2 & \\ & \beta_2 & \alpha_3 & \\ & & \beta_3 & \end{bmatrix}$$

$\alpha_1 = \eta_1^2, \Rightarrow$

$\beta_2 = \eta_1 \nu_2 =$

$\alpha_2 = \nu_2^2 + \eta_2$

$\beta_3 = \eta_2 \nu_3 \Rightarrow$

ho Read

$$\begin{bmatrix} & & & \\ & & & \\ \beta_3 & & & \\ \alpha_4 & & & \end{bmatrix} = \begin{bmatrix} \eta_1 & & & \\ \mu_2 & \eta_2 & & \\ & \mu_3 & \eta_3 & \\ & & \mu_4 & \eta_4 \end{bmatrix} \begin{bmatrix} \eta_1 & \mu_2 & & \\ & \eta_2 & \mu_3 & \\ & & \eta_3 & \mu_4 \\ & & & \eta_4 \end{bmatrix}$$

$\Rightarrow \eta_1 = \sqrt{\alpha_1}$

$\Rightarrow \mu_2 = \beta_2 / \eta_1$

$2 \quad \Rightarrow \eta_2 = \sqrt{\alpha_2 - \mu_2^2}$

$\Rightarrow \mu_3 = \beta_3 / \eta_2 \quad \ldots$

$k \to k+1$

$\mu_k = \beta_k / \eta_{k-1}$

$\eta_k = \sqrt{\alpha_k - \mu_k^2}$

Pf 2: Induction

Base Case

$\alpha_1 = \eta_1^2 = \checkmark$

Assume $F_k$

$$T_k = \begin{bmatrix} \eta_1 & & & \\ \mu_2 & \ddots & & \\ & \ddots & & \\ & & \mu_k & \eta_k \end{bmatrix} F_k^T$$

We want to show

$$T_{k+1} = \begin{bmatrix} T_k & \beta_{k+1} e_k \\ \beta_{k+1} e_k^T & \alpha_{k+1} \end{bmatrix}$$

$$F_k F_k^T = \bar{T}_k$$ ✓

$$\beta_{k+1} \underline{e}_k = \underbrace{F_k \underline{e}_k}_{= \eta_k \underline{e}_k} (\mu_{k+1})$$

$$\beta_{k+1} \underline{e}_k = \eta_k \mu_{k+1} \underline{e}_k \Rightarrow$$

$$\mu_{k+1} = \beta_{k+1} / \eta_k$$ ✓

$$\alpha_{k+1} = \mu_{k+1}^2 + \eta_{k+1}^2$$

$$\Rightarrow \eta_{k+1} = \sqrt{\alpha_k - \mu_{k+1}^2}$$

$$] = \overline{\begin{bmatrix} F_k & 0 \\ \mu_{k+1} \underline{e}_k^T & \eta_{k+1} \end{bmatrix}} \begin{bmatrix} F_k^T & \mu_{k+1} \underline{e}_k \\ 0 & \eta_{k+1} \end{bmatrix}$$

CG, Goal

$x_k = V_k y$

where $\bar{T}_k y_k$

We have (@

$\bar{T}_k \longrightarrow$

$F_k \longrightarrow$

what about

$y_k \to y_{k+1}$

$k$

$= \|b\| e_1$

step $k$)

we can get (fast!)

$\alpha_1$ —— $\alpha_k$ $\quad\alpha_{k+1}$

$\beta_2$ —— $\beta_k$ $\quad\beta_{k+1}$

$\to$

$\eta_1 \to \eta_k$ $\quad\eta_{k+1}$

$\nu_2 \to \nu_k$ $\quad\nu_{k+1}$

?

@ step $k$ we want soln

$$F_k \underbrace{F_k^T y_k}_{z_k} = \|b\| e_1$$

$$\downarrow$$

$$F_k z_k = \|b\| e_1 \longleftrightarrow$$

So we can update

$$z_k \to z_{k+1}$$

@ step

$$- \begin{bmatrix} F_k^T \\ \beta_{k+1} e_k \end{bmatrix}$$

$$\begin{bmatrix} F_k \\ \nu_{k+1} e_k^T \end{bmatrix}$$

$\nu_{k+1} e_k^T$

$z_{k+1}$

$_{k+1}$ = Scalar.

$$\begin{bmatrix} \beta_{k+1} e_k \\ \alpha_{k+1} \end{bmatrix} y_{k+1} = \begin{bmatrix} \|b\| e_1 \\ 0 \end{bmatrix}$$

Is this $\underline{z}_k$? Yes!

$$\begin{bmatrix} 0 \\ \eta_{k+1} \end{bmatrix} \begin{bmatrix} \underline{z}_k \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} \|b\| e_1 \\ 0 \end{bmatrix}$$

$$^T \underline{z}_k + \eta_{k+1} z_{k+1} = 0$$

$$= \frac{\mu_{k+1} \cdot z_k}{-\eta_{k+1}}$$

Now $F_k^T y_k = \underline{z}_k$

Can we get $y_k \to y_{k+1}$?

$$\begin{bmatrix} F_k^T & \mu_{k+1} \underline{e}_k \\ 0 & \eta_{k+1} \end{bmatrix} y_{k+1} = \begin{bmatrix} \underline{z}_k \\ z_{k+1} \end{bmatrix}$$

$$[y_{k+1}]_{k+1} = \frac{z_{k+1}}{\eta_{k+1}}$$

$$[y_{k+1}]_k = \frac{z_k - \mu_{k+1} [y_{k+1}]_{k+1}}{\eta_k}.$$

$\vdots$

So all $y_k$ change from $k \to k+1$

But, we don't want $y_k$!

We want $x_k$!

$$x_k = V_k y_k$$

$$= \|b\| V_k T_k^{-1} \underline{e}_1$$

$$= \underbrace{V_k F_k^{-T}}_{= C_k} \underbrace{F_k^{-1} \|b\| \underline{e}_1}_{= \underline{z}_k}$$

$C_k = V_k F_k^{-1}$

Can we go $C_k \to C_{k+1}$?

$$C_k = V_k F_k^{-T}$$

$$\Longleftrightarrow$$

$$V_k = C_k F_k^T$$

Assume we have $\underline{C}_k$

Let $C_{k+1} = [D \;\; \underline{g}]$

$$V_{k+1} = C_{k+1} \cdot F_{k+1}^T$$

$$\Longleftrightarrow$$

$$[V_k \;\; \underline{V}_{k+1}] = [D \;\; \underline{g}] \begin{bmatrix} F_k^T & \mu_{k+1} \underline{e}_k \\ 0 & \eta_{k+1} \end{bmatrix}$$

$$= [D F_k^T \quad \mu_{k+1} D$$

$$\Longleftrightarrow$$

$$D F_k^T = V_k \longrightarrow$$

$$\mu_{k+1} D \underline{e}_k + \eta_{k+1} \underline{g} = \underline{V}_{k+1}$$

$$\underline{g} = \frac{\underline{V}_{k+1} - \mu_{k+1} \boxed{\underline{C}}}{\eta_{k+1}}$$

$$X_k = \underline{C}_k \underline{z}_k$$

$$X_{k+1} = [C_k \;\; \underline{c}_{k+1}]$$

$$= \underbrace{C_k \underline{z}_k}_{= X_k} + \underline{C}_{k+1}$$

$\underline{e}_k + \eta_{k+1} \underline{g} \Big)$

$= C_k$ . ! ☺

last col of $C_k$! (called $c_k$)

$e_k$

$= C_{k+1}$

$\begin{bmatrix} \underline{z}_k \\ z_{k+1} \end{bmatrix}$

$z_{k+1}$

er Science - David F. Gleich - 2019 CS 515 - Purdue University - Computer Science - David F. Gleich - 2019 CS 515 - Purdue University - Computer Science - David F. Gleich - 2019 CS