Lecture 28

- Subspace iteration: get all the eigenvalues & vectors at once!
- The QR method is subspace iteration started from the identity.
- We can make this more efficient by reducing the matrix to a tridiagonal matrix first.
- Reduce the number of iterations? See the books. Shift and invert.
- Eigenvalues of sparse matrices. How we can use the Lanczos method to get good estimates.
- How ARPACK works. (The sparse solver in Matlab and Julia.)

HW 11 - Multigrid function

$\div$ = div

Subspace iteration

$X^{(0)}$ = random matrix.

for $k = 0$ to ...

$Y = AX^{(0)}$

$Q, R = qr(Y)$

$X^{(k+1)} = Q$

Power Method

$x^{(0)}$ = random

for $k = 0$ to ...

$y = Ax^{(k)}$

$x^{(k+1)}$ = normalize(y)

Many vec $\Longleftrightarrow$ 1 vector

$X^{(0)} = I$

Let's adjust notation!

Subspace

Given A

$X_0 = I$

for $i = 1$ to ...

$Y = AX_{i-1}$

$X_i R_i = qr(Y)$

QR

Given A

Let $A_1 = A$

for $i = 1$ to ...

$Q_i, R_i = qr(A)$

$A_{i+1} = R_i Q_i$

Note! Remember

$R_i = Q_i^T A_i$ so

$R_i Q_i = Q_i^T A_i Q_i$

=> So each A_i has the same λ-vals.

Lemma

$X_i = Q_1 \cdots Q_i$

Base Case: Subspace

$X_1 = qr(AX_0)$

$= qr(A)$

$Q_1 = qr(A_1) = qr(A)$ QR

(Done w/ Base!)

Induction Hypoth.

$X_i = Q_1 \cdots Q_i$

Subspace

$Y = AX_i$

$qr(Y) = qr(AQ_1 \cdots Q_i)$

$X_{i+1} = qr(Y) \Leftarrow$

$R_{i+1} =$

$\boxed{X_{i+1}^T} A Q_1 \cdots Q_i$

Need to show that

$X_{i+1} = Q_1 \cdots Q_{i+1}$

$Q_{i+1}^T \cdots Q_1^T A Q_1 \cdots Q_i$ = upper-tri

$\Rightarrow X_{i+1} = Q_1 \cdots Q_{i+1}$

QR

$A_{i+1} = Q_i^T \overset{A_i}{A_i} Q_i$ at step i

so

$A_{i+1} = \underbrace{Q_i^T Q_{i-1}^T \cdots Q_1^T}_{} A \underbrace{Q_1 \cdots Q_i}_{}$

$A_{i+1} = X_i^T A X_i$

$Q_{i+1} R_{i+1} = qr(X_i^T A X_i)$

$R_{i+1} = Q_{i+1}^T X_i^T A X_i$

(upper-tri)

=>

(by "uniqueness of QR")

Also: $A_i = X_i^T A X_i$

=> diag matrix w/ λ-vals.

QR w/ Eigenvectors

Let $A_1 = A$

for $i = 1$ to ...

$Q_i R_i = qr(A_i)$

$A_{i+1} = R_i Q_i$

$X \Leftarrow X \cdot Q_i$

What to improve?

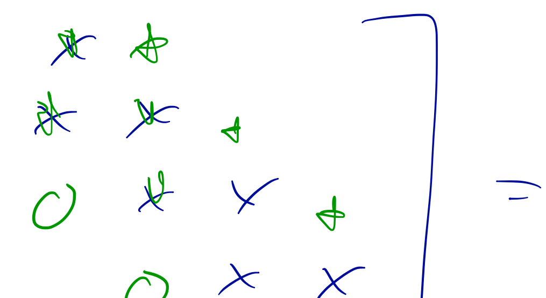
$O(n^3)$ per-iter ☹

If A were tridiag. then $qr(A)$ can be done with

Quiz!

Givens Rotation!

If A is tridiag. (Sym.)



$= R$

$$Q_1 = \begin{bmatrix} x & x & & & \\ x & x & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 1 & & & \\ & x & x & \\ & x & x & \\ & & & 1 \end{bmatrix}$$

Slick proof that RQ is tridiag.

A = tridiag, sym.

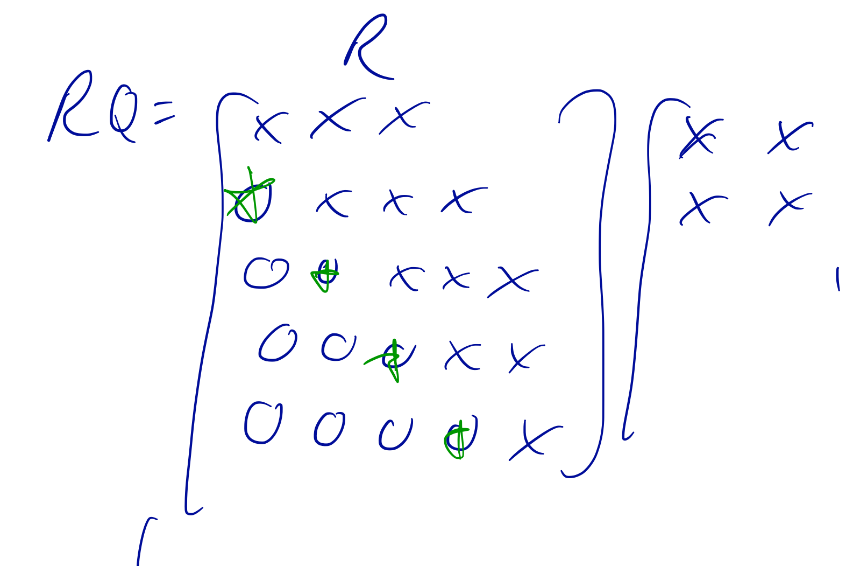
$QR = A \Rightarrow R = Q^T A$

$RQ = Q^T A Q$

= Sym.

=> RQ is tridiag. if RQ is upper Hessenberg.

$RQ =$



lin comb of 1, 2

Last prob: How to make Sym. A into tridiag A and keep λ-val info (and Eigenvectors too.)

find

$X^{-1} A X$ = tridiag or

$Q^T A Q$ = tridiag.

for orthog. Q.

=> Use Lanczos!

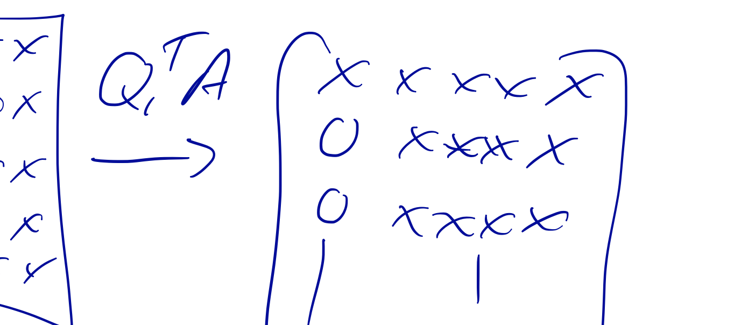
Instead we will use Householder!

(A is Sym.)

$A \to Q^T A Q$ = tridiag

Idea 1:

A



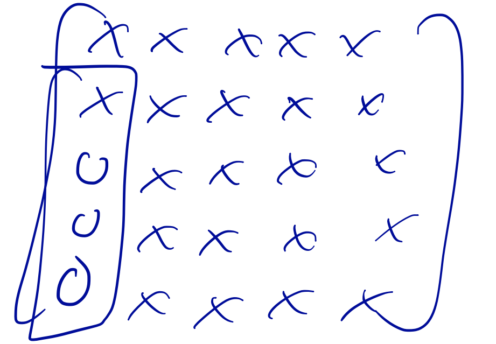
Do same on right

$Q_1^T A Q_1 = ?$

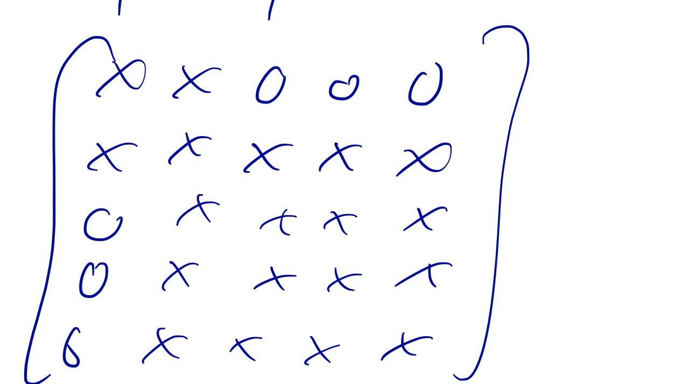
destroys the zeros ☹

Idea 2: be less greedy!

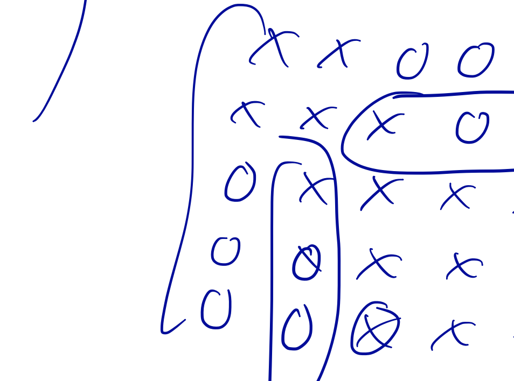
$Q_1^T A =$



$Q_1^T A Q_1$



$Q_2^T Q^T A Q_1 Q_2 =$



$Q_3^T Q_2^T Q_1^T A Q_1 Q_2 Q_3 =$

tridiag! ☺

How ARPACK works?

We want λ's of A.

Run Lanczos for

$p > k$ steps

"or more"

$A \to V_p, T_p$ - tridiag

Extreme λ's of A are

λ's of T_p for $\begin{bmatrix} T_p \\ \beta_p e_p^T \end{bmatrix}$

$p > k$

V_p

↓ Implicit Restart

$\boxed{V_k^{(1)}}$

$T_k^{(1)}$

↓ grow

$V_p^{(1)}$

$T_p^{(1)}$

↓ shr