LECTURE NOTES
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Let us begin by introducing basic notation for matrices and vectors.

Matrices

We'll use $\mathbb{R}$ to denote the set of real-numbers and $\mathbb{C}$ to denote the set of complex numbers.

We write the space of all $m \times n$ real-valued matrices as $\mathbb{R}^{m\times n}$. Each

$$\boldsymbol{A} \in \mathbb{R}^{m\times n} \text{ is } \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{bmatrix} \text{ where } A_{i,j} \in \mathbb{R}.$$

Sometimes, I'll write:

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix}$$

instead. With only a few exceptions, matrices are written as *bold, capital* letters. Sometimes, we'll use a capital greek letter. Matrix elements are written as subscripted, *unbold* letters.
When clear from context,

$$A_{i,j} \text{ is written } A_{ij}$$

instead, e.g. A_{11} instead of $A_{1,1}$.

In class I'll usually write matrices with just upper-case letters. If you are unsure if something is a matrix or an element, raise your hand and ask, or *quietly* ask a neighbor.

Another notation for $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is

$$\boldsymbol{A} : n \times n.$$

Sometimes this is nicer to write on the board.

Vectors

We write the set of length-n real-valued vectors as $\mathbb{R}^n$. Each

$$\mathbf{x} \in \mathbb{R}^n \text{ is } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ where } x_i \in \mathbb{R}.$$

Vectors are denoted by *lowercase, bold* letters. As with matrices, elements are sub-scripted, *unbold* letters. Sometimes, we'll write vector elements as

$$x_i \text{ or } [x]_i \text{ or } x(i).$$

Usually, this choice is motivated by a particular application. *Throughout the class, vectors are* ***column*** *vectors.*

In class I'll usually write vectors with just lower-case letters and *will try* to follow the convention of underlining them.

Scalars

Lower-case greek letters are scalars.

Quick test

Identify the following:

$$\mathbf{f}, z_1, \mathbf{x}_1, \alpha, \beta, \boldsymbol{C}, \boldsymbol{C}_1, \boldsymbol{\Sigma}, B_{i,j}, \mathbf{b}_{i,j}$$

Operations

Transpose Let $\boldsymbol{A} : m \times n$, then

$$\boldsymbol{B} : n \times m = \boldsymbol{A}^T \text{ has } B_{i,j} = A_{j,i}.$$

Example $\boldsymbol{A} = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 4 \\ 3 & -1 \end{smallmatrix}\right]$ $\quad \boldsymbol{A}^T = \left[\begin{smallmatrix} 2 & 1 & 3 \\ 3 & 4 & -1 \end{smallmatrix}\right]$

Hermitian (Also called conjugate transpose) Let $\boldsymbol{A} \in \mathbb{C}^{m \times n}$, then

$$\boldsymbol{B} \in \mathbb{C}^{n \times m} = \boldsymbol{A}^* = \boldsymbol{A}^H \text{ has } B_{i,j} = \overline{A}_{j,i}.$$

Example $\boldsymbol{A} = \left[\begin{smallmatrix} 2 & 3 & \\ i & 4 & \\ & 3 & -i \end{smallmatrix}\right]$ $\quad \boldsymbol{A}^* = \left[\begin{smallmatrix} 2 & -i & 3 \\ 3 & 4 & i \end{smallmatrix}\right]$

Addition Let $\boldsymbol{A} : m \times n$ and $\boldsymbol{B} : m \times n$, then

$$\boldsymbol{C} : m \times n = \boldsymbol{A} + \boldsymbol{B} \implies C_{i,j} = A_{i,j} + B_{i,j}.$$

Example $\boldsymbol{A} = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 4 \\ 3 & -1 \end{smallmatrix}\right], \boldsymbol{B} = \left[\begin{smallmatrix} 1 & -1 \\ 2 & 3 \\ -1 & 1 \end{smallmatrix}\right]$ $\boldsymbol{A} + \boldsymbol{B} = \left[\begin{smallmatrix} 3 & -2 \\ 3 & 2 \\ 2 & 0 \end{smallmatrix}\right].$

Scalar Multiplication Let $\boldsymbol{A} : m \times n$ and $\alpha \in \mathbb{R}$, then

$$\boldsymbol{C} : m \times n = \alpha \boldsymbol{A} + \boldsymbol{B} \implies C_{i,j} = \alpha A_{i,j}.$$

Example $\boldsymbol{A} = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 4 \\ 3 & -1 \end{smallmatrix}\right], 5\boldsymbol{A} = \left[\begin{smallmatrix} 10 & 15 \\ 5 & 20 \\ 15 & -5 \end{smallmatrix}\right]$

Matrix Multiplication Let $\boldsymbol{A} : m \times n$ and $\boldsymbol{B} : n \times k$, then

$$\boldsymbol{C} : m \times k = \boldsymbol{A}\boldsymbol{B} \implies C_{i,j} = \sum_{r=1}^{n} A_{i,r} B_{r,j}.$$

Matrix-vector Multiplication Let $\boldsymbol{A} : m \times n$ and $\mathbf{x} \in \mathbb{R}^n$, then

$$\mathbf{c} \in \mathbb{R}^m = \boldsymbol{A}\mathbf{x} \implies c_i = \sum_{j=1}^{n} A_{i,j} x_j.$$

This operation is just a special case of matrix multiplication that follows from treating $\mathbf{x}$ and $\mathbf{c}$ as $n \times 1$ and $m \times 1$ matrices, respectively.

Vector addition, **Scalar vector multiplication** These are just special cases of matrix addition and scalar matrix multiplication where vectors are viewed as $n \times 1$ matrices.

Partitioning

It is often useful to represent a matrix as a collection of vectors. In this case, we write

$$\boldsymbol{A} : m \times n = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

where each $\mathbf{a}_j \in \mathbb{R}^m$. *This form corresponds to a partition into columns.*

Alternatively, we may wish to partition a matrix into rows.

$$\boldsymbol{A} : m \times n = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix}.$$

Here, each $\mathbf{r}_i \in \mathbb{R}^n$.

Using the column partitioning:

$$\boldsymbol{A}\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_j x_j \mathbf{a}_j.$$

And with the row partitioning:

$$\boldsymbol{A}\mathbf{x} = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{r}_1^T \mathbf{x} \\ \mathbf{r}_2^T \mathbf{x} \\ \vdots \\ \mathbf{r}_m^T \mathbf{x} \end{bmatrix}.$$

Another useful partitioned representation of a matrix is into blocks:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{1,1} & \boldsymbol{A}_{1,2} \\ \boldsymbol{A}_{2,1} & \boldsymbol{A}_{2,2} \end{bmatrix}$$

or

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{1,1} & \boldsymbol{A}_{1,2} & \boldsymbol{A}_{1,3} \\ \boldsymbol{A}_{2,1} & \boldsymbol{A}_{2,2} & \boldsymbol{A}_{2,3} \\ \boldsymbol{A}_{3,1} & \boldsymbol{A}_{3,2} & \boldsymbol{A}_{3,3} \end{bmatrix}.$$

Here, the sizes "just have to work out" in the vernacular. Formally, all $\boldsymbol{A}_{i,\cdot}$ must have the same number of rows and all $\boldsymbol{A}_{\cdot,j}$ must have the same number of columns.