Lecture 11

HW 5 posted!

Handout!

I'm gone Thursday.

- Duality again (no typos)

The geometry of LPs!

- feasible polytope
- vertices.

The fundamental theorem of LP: A soln must exist at a vertex of the feasible polytope.

Pf?!

$$\min \ \underline{c}^T \underline{x} \quad \text{s.t.} \quad A\underline{x} = \underline{b}, \quad \underline{x} \ge 0$$

$$\mathcal{L}(\underline{x}, \underline{\lambda}, \underline{s}) = \underline{c}^T \underline{x} - \underline{\lambda}^T (A\underline{x} - \underline{b}) - \underline{x}^T \underline{s}$$

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① $\underline{s} \ge 0$

② $\underline{x}^T \underline{s} = 0$

③ $A\underline{x} = \underline{b}$

④ $\underline{x} \ge 0$

⑤ $\underline{c} - A^T \underline{\lambda} - \underline{s} = 0$

$$\max \ \underline{b}^T \underline{\lambda} \quad \text{s.t.} \quad A^T \underline{\lambda} \le \underline{c}$$

$$\Downarrow$$

$$\min \ -\underline{b}^T \underline{\lambda} \quad \text{s.t.} \quad A^T \underline{\lambda} \le \underline{c}$$

$$\mathcal{L}(\underline{\lambda}, \underline{x}) = -\underline{b}^T \underline{\lambda} - \underline{x}^T (\underline{c} - A^T \underline{\lambda})$$

Opt. Condtion.

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\underline{b} + A\underline{x} =$$

$$A^T \underline{\lambda} \le \underline{c}$$

$$\underline{x} \ge 0$$

$$\underline{x}^T (\underline{c} - A^T \underline{\lambda}) = 0$$

Claim

Red <=> Green.

Why? if

$$\underline{c} - A^T\lambda = s \Longleftrightarrow$$

$$\underline{c} - A^T\underline{\lambda} - \underline{s} = 0$$

and $\underline{s} \ge 0$ since

$$A^T\lambda \le c$$

Then we add.

⓪ $c - A^T\lambda = s$

① $\to A x = \underline{b}$

② $\to \underline{s} \ge 0$

③ $\to \underline{x} \ge 0$

④ $\to x^T \underline{s} = 0.$

Geometry of LPs

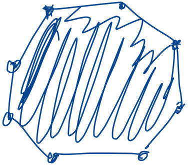
Consider an LP in standard form

$$\min \ \underline{c}^T \underline{x}$$
$$\text{s.t.} \ \underline{A}\underline{x} = b$$
$$\underline{x} \ge 0.$$

The feasible polytope is the set

$$F = \{\underline{x} \mid A\underline{x} = \underline{b}, \underline{x} \ge 0\}$$

A polytope is a higher dim version of a polygon



A Common Example

$$\min \; -x_1 - 2x_2$$

$$\text{s.t. } x_1 \le 3$$

$$-x_1 + 2x_2 \le 7$$

$$-2x_1 + x_2 \le 2$$

$$x_1, x_2 \ge 0.$$

In standard form, add

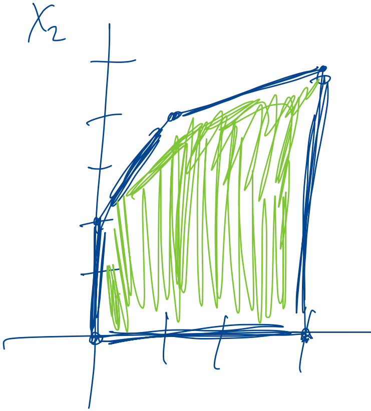
$$s_1, s_2, s_3 \ge 0$$

$$\text{s.t. } x_1 + s_1 = 3$$

$$-x_1 + 2x_2 + s_2 = 7$$

$$-2x_1 + x_2 + s_3 = 2$$

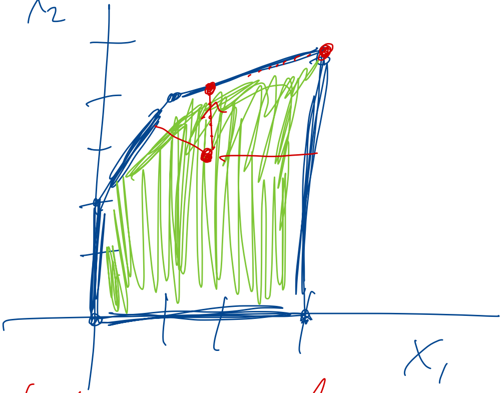
The feasible region is



Where are the sla

x_1, x_2 is

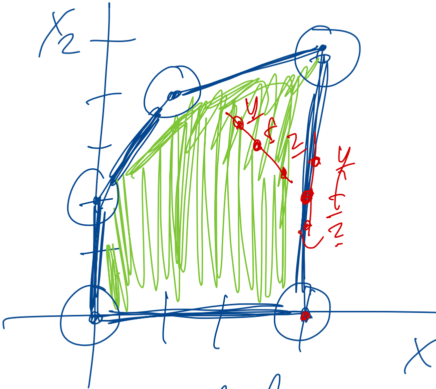
ks ?



Slacks are just distances to constraints.

If you view $\begin{bmatrix} x \\ s \end{bmatrix}$ as a joint vector, then at a boundary you get a new zero value.

Vertices



A vertex of the feasible poly are points $f \in F$ where f cannot be written as a convex combo of any other points $y, z \in F$ (cy

tope

$\neq z)$

Any point that is not a vertex can be written as

$$\underline{f} = \alpha y + (1-\alpha) \underline{z}$$

$$0 < \alpha, 1-\alpha, < 1$$

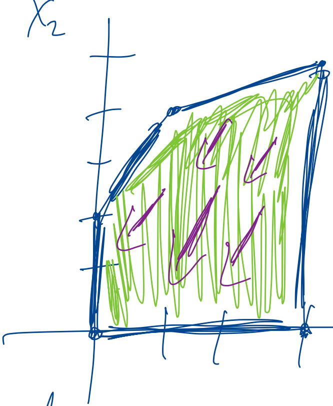
for $y \neq \underline{z}$.

The fundamental theorem of LPs

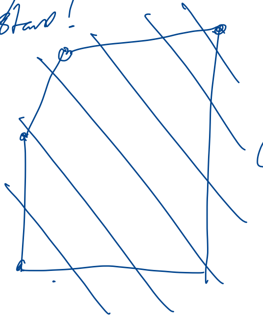
If an LP has a solution, then it must occur at a vertex of the feasible polytope.

Quiz: Why is this "obvious"?

Hint: think about the gradient!



The gradient is constant!



Pf of theorem

(‡) Our proof is in the bounded case (‡)

1. Any feasible point can be written as a convex combo of vertices of F.

$$\underline{x} \ge 0, \; A\underline{x} = \underline{b} \iff$$

$$\underline{x} = \sum_{i=1}^{|V|} \alpha_i \underline{v}_i$$

s.t. $\sum \alpha_i = 1, \; \alpha_i \ge 0.$

2. If that's true then

$$\underline{c}^T \underline{x} = \sum_{i=1}^{|V|} \alpha_i \underline{c}^T \underline{v}_i$$

Let $\underline{v}^*$ be the vector that minimize $c^T v_i$.

i.e. $v^* = \operatorname{argmin}_{i=1}^{|V|} c^T \underline{v}_i$

$$\underline{c}^T \underline{x} \ge \sum_{i=1}^{|V|} \alpha_i \underline{c}^T \underline{v}^*$$

$$= \underline{c}^T \underline{v}^*.$$

Pf of $\underline{x}$ is feasible $\Rightarrow$

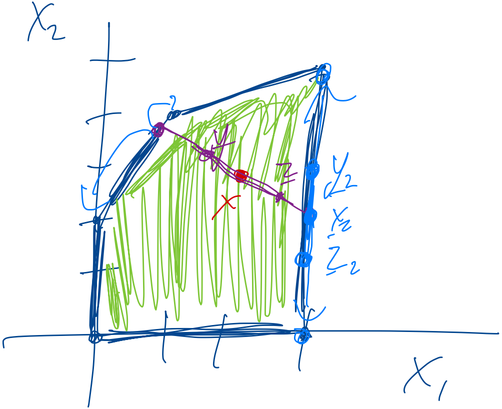
$$\underline{x} = \sum_{i \in \text{Vertex}} \alpha_i \underline{v}_i \quad (\text{Convex})$$

Let $\underline{x}$ be feasible, then

Either $\underline{x}$ is a vertex or

I can find $\underline{y}, \underline{z}$ s.t.

$$\alpha \underline{y} + (1-\alpha)\underline{z} = \underline{x}.$$



Let p be the line $y-z$

Note, $A\underline{p} = A(y-z)$

$= Ay - Az$

$= \underline{b} - \underline{b} = 0.$

$y_i \underline{z} \geq 0$.

I claim I can always find such a $\underline{p}$ w/ the same # of non-zero entries as $\underline{x}$.

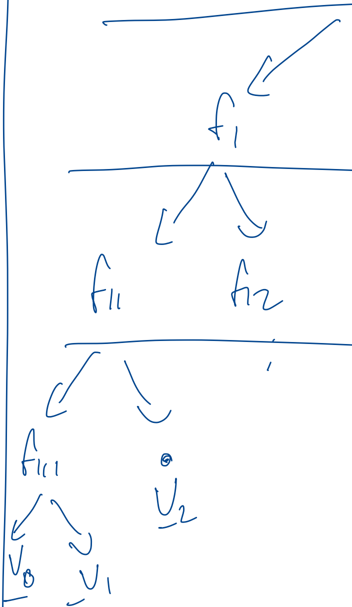
Idea: we find $\gamma_1 > 0$, $\gamma_2 < 0$

$f_1 = \underline{x} + \gamma_1 \underline{p}$

$f_2 = x + \gamma_2 p$

have one more zero component than $\underline{x}$.

Then repeat on $\underline{f_1}$, $\underline{f_2}$ w/ one fewer non-zero!



X (n non-zero

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f_2 (n-1 non-zero)

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n-2

⋮