Lecture 12

Definition: Basic Feasible Points

Result 1: BFPs are useful characterizations of vertices

LP Alg 1: Just check all BFPs!

Result 2: We can get dual variables from a BFP. This means we can locally certify a max point!

Result 3: Given a BFP and the dual variables, if the BFP is not optimal, then we can find a new BFP that has a lower objective function.

LP Alg 2: Use result 3 over and over again! This is the simplex method.

$$\min c^T x \quad \text{s.t.} \quad Ax = b, \; x \geq 0$$

KKT Condition

$$S \geq 0$$
$$x \geq 0$$
$$Ax = b$$
$$x^T S = 0$$
$$c = A^T \lambda + S$$

Lagrange form of an LP has a lot of duality then.

Dual Ordering suff to illustrate

A Basic Feasible Point (BFP) is a feasible pnt

$$Ax = b, \; x \geq 0$$

Where number $|A^T \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b|$

x_B is non-zero.

$$A^T x = \begin{bmatrix} A_B \\ 0 \end{bmatrix} = \begin{bmatrix} x_B \\ 0 \end{bmatrix}$$

A BFP has zeros in x corresponding to columns of A.

Exactly m of these don't cut it

A BFP $\iff$ m columns of A.

Stack up m columns of A where A_B is non-sing. then, x calc

$$A^T x = \begin{bmatrix} A_B \\ 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} A_B \\ 0 \end{bmatrix}$$

We can write $x = A_B^{-1} b$

$$Ax = [A_B \; A_N] \begin{bmatrix} A_B \\ 0 \end{bmatrix} = A_B A_B^{-1} b = b$$

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function isBFP(A,b,C)
# C is a set of m numbers for columns of A.
B = A[:,C] extract the columns
if rank(B) != size(A,1) # check full rank
    return false
else
    xb = B\b # solve a set associated with C cols
    if all(xb >= 0)
        # Build the vertex coordinate
        x = zeros(size(A,2))
        x[C] = xb
        # we don't need to set x[C] = 0
        # because x = zeros initially
        return true, x
    else
        return false
    end
end
```

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$C = [3, 4, 1]$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$C = [5, 3, 4]$$

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Example

Vertices

BFP $\iff$ Vertex

Case 1: BFP $\Rightarrow$ Vertex

Case 2: Vertex $\Rightarrow$ BFP

Case 1: If x is a vertex we want to show that

$$\exists \; C \ni x = C$$

$$x = \begin{bmatrix} x_B \\ 0 \end{bmatrix} \quad \text{where} \quad x_B = A_B^{-1} b$$

We show this with the Σ $\begin{bmatrix} x_B \\ 0 \end{bmatrix}$ complete solution.

Result 1

BFP $\iff$ Vertex

Case 1: BFP $\Rightarrow$ Vertex

Case 2: Vertex $\Rightarrow$ BFP

Case 1: If x is a vertex we want to show that

$$\exists \; C \ni x = C$$

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We show this with the Σ $\begin{bmatrix} x_B \\ 0 \end{bmatrix}$ complete solution.

$$A_B x = [A_B \; A_N] \begin{bmatrix} x_B \\ 0 \end{bmatrix} = A_B A_B^{-1} b = b$$

$$x = \begin{bmatrix} x_B \\ 0 \end{bmatrix} \quad \text{is} \quad A^T x = \begin{bmatrix} A_B \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_B \\ y_N \end{bmatrix} = \begin{bmatrix} y_B \\ 0 \end{bmatrix}$$

Also $y_N \geq 0$ $\iff$ $y_B \geq 0$.

$$A_B^T y_B = A_B^T \lambda$$

$$\begin{bmatrix} A_B \\ A_N \end{bmatrix} [A_B \; A_N] \begin{bmatrix} y_B \\ 0 \end{bmatrix} = \begin{bmatrix} A_B \\ A_N \end{bmatrix} A_B^T \lambda$$

$$\begin{bmatrix} A_B A_B^T & A_B A_N^T \\ A_N A_B^T & A_N A_N^T \end{bmatrix} \begin{bmatrix} y_B \\ 0 \end{bmatrix} = \begin{bmatrix} A_B A_B^T \\ A_N A_B^T \end{bmatrix} \lambda$$

$$\begin{bmatrix} A_B A_B^T y_B \\ A_N A_B^T y_B \end{bmatrix} = \begin{bmatrix} A_B A_B^T \lambda \\ A_N A_B^T \lambda \end{bmatrix}$$

$$A_N A_B^T y_B = A_N A_B^T \lambda$$

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$$A_N A_B^T y_B = A_N A_B^T \lambda