Lecture 13

Midterm Class before Spring Break. March 13th.

Recap of Simplex Method and LPs (10-15 mins)

- BFP
- x, lambda, x from BFP
- find a new BFP
- show new BFP has a smaller objectives

Phase 1 or Crash to find an initial BFP

Runtime of Simplex for LPs (no good news)

— Done with LPs —

Back to unconstrained opt.

Two types of methods

- Line search
- Trust region

The general line-search setup

Definition: a descent direction

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HW6 posted

Grab a handout! ← #4

$$\min \ \underline{c}^T \underline{x}$$

$$\text{s.t.} \quad A\underline{x} = \underline{b} \qquad (m \times n)$$

$$\underline{x} \ge 0 \qquad m \le n$$

Theorem a soln of an LP always exists at a vertex of the feasible polytope.

KKT:

$$\underline{x} \ge 0$$

$$\underline{x}^T \underline{s} = 0$$

$$\underline{s} \ge 0$$

$$A\underline{x} = \underline{b}$$

$$\underline{c} = A^T \underline{\lambda} + \underline{s}$$

BFP $\Leftrightarrow$ vertices of the feasible polytope.

BFPs is a set of columns C s.t.

$$\boxed{B = A(:, C) \text{ is non-sing. and}}$$

$$\underline{x} = \begin{bmatrix} B^{-1}\underline{b} \\ 0 \end{bmatrix} \Leftrightarrow$$

$$B^{-1}b \ge 0 \ \checkmark$$

✓

$\begin{bmatrix} x_B \\ x_N \end{bmatrix}$

Given a BFP, we can get estimates of $\underline{\lambda}$, $\underline{s}$

Let $\underline{x} = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$, $c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}$

$\underline{s} = \begin{bmatrix} s_B \\ s_N \end{bmatrix}$. Assume $s_B = 0$.

$$\begin{bmatrix} c_B \\ c_N \end{bmatrix} - \begin{bmatrix} B^T \\ N^T \end{bmatrix} \underline{\lambda} - \begin{bmatrix} 0 \\ \underline{s}_N \end{bmatrix} = 0.$$

$$\lambda = B^{-T} \underline{c}_B$$

$$\underline{s}_N = \underline{c}_N - N^T \underline{\lambda}$$

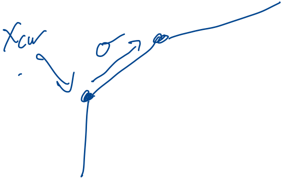
If $s_N \geq 0 \Rightarrow$ OPT soln because KKT satisfied!

So if $(s_N)_j < 0$, for some j, we need more work ☺.

What SIMPLEX does is use

$$x^{(next)} = \begin{bmatrix} x_B^{(next)}(\sigma) \\ \sigma e_j \end{bmatrix}$$

Chosen to stay feasible.



SIMPLEX moves along faces/Edges until it hits a new vertex.

$x_B^{next}(\sigma) =$

… Easy to work out from feasibility …

$$[B \;\; N] \begin{bmatrix} x_B^{(next)}(\sigma) \\ \sigma e_j \end{bmatrix} = \underline{b}$$

$$B x_B^{(next)} \sigma + \sigma N e_j = \underline{b}$$

$$x_B^{(next)} = \underbrace{B^{-1} \underline{b}}_{x_B} - \sigma \underbrace{B^{-1} N e_j}_{= d}.$$

$$x_B^{(next)}(\sigma) = x_B - \sigma \underline{d}.$$

Recall $\underline{x}_B \ge 0$

We need $d_i \ge 0$ to possibly bump into a new point.

if $d_i > 0$, then solve

$$[x_B]_i - \sigma d_i = 0 \Rightarrow$$

$$\sigma_i = \frac{[x_B]_i}{d_i} \quad (d_i > 0)$$

find smallest σ_i.

Remove column i from B, add column j.

Result New

$$\underline{c}^T \underline{x}^{(next)} < \underline{c}^T \underline{x}^{(cur)}$$

$$\|$$

$$\begin{bmatrix} c_B \\ c_N \end{bmatrix}^T \begin{bmatrix} x_B^{next}(\sigma) \\ \sigma \underline{e}_j \end{bmatrix} =$$

$$\begin{bmatrix} c_B \\ c_N \end{bmatrix}^T \begin{bmatrix} x_B - \sigma B^{-1} N e_j \\ \sigma e_j \end{bmatrix}$$

$$= \underbrace{c_B^T x_B}_{= c^T \underline{x}^{(cur)}} - \sigma c_B^T B^{-1} N e_j + \sigma c_N^T \underline{e}_j$$

=

I

S

Note

$c^T x^{(cur)}$

$- \sigma (c_B^T B^{-1} N - c_N) e_j$

It'd be nice if

s_N is Related to

$c_B^T B^{-1} N - c_N$!

$s_N = c_N - N^T \lambda$

$\lambda = B^{-T} c_B$

So $c_B^T B^{-1} N - c_N^T = -s_N^T$.

$(s_N)_j < 0$!

$c^T x^{(next)} =$

$c^T x^{(cur)} - \sigma(-s_N^T) e_j$

$= c^T x^{(cur)} + \sigma \underbrace{s_N^T e_j}_{<0}$

But $(s_N)_j < 0$, so..

$= c^T x^{(cur)} + \sigma [\text{neg}]$

$< c^T x^{(cur)}$

if $\sigma > 0$.

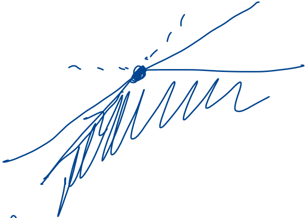
$\Rightarrow$ always get a ~~Smaller~~ vertex w/ a smaller objective.

Thus SIMPLEX
Converges to a minimizer.

Cases to handle.

1. $(S_N)_j = 0 \Rightarrow$ already @ KKT.
$\Rightarrow$ soln

What if $\sigma = 0$?



degenerate constraints
redundant) are possible.

Need to pick
j (a column of
A) consistently to
Solve.
$\Rightarrow$ See anti-
cycling rule.

③ Is $x^{(next)}(\sigma)$
a vertex?

Feasible ✓ (by
Construction)

m-non-zeros ✓
(by construction)

Non-sing.?

Adding Ne_j
(New column).

Is this a lin.
comb of B?
(What's left in B?)

(Simple Example)
Suppose $N\underline{e}_j = B\underline{e}_k$
for some k.
Then
$$\underline{d} = B^{-1}N\underline{e}_j = B^{-1}B\underline{e}_k$$
$$= \underline{e}_k$$

So we MUST kick out
column k!

More generally, if adding a column would make the next set of cols sing, then $d_i > 0$ for those cols we could remove so this doesn't happen.

=> Done!

Almost.

We assume we start at a BFP! Yikes!

How to get one BFP??

Idea: Solve an LP where we know a BFP to get a BFP for our LP.

$$\min_{\underline{x}, \underline{z}} \quad \underline{e}^T \underline{z} = \Sigma$$

$$\text{s.t.} \quad [A \;\; I] \begin{bmatrix} \underline{x} \\ \underline{z} \end{bmatrix}$$

$$\underline{x}, \underline{z} \ge 0.$$

(Assume $\underline{b} \ge 0$)

What does a soln of this LP look like?

Quiz!

z_i -

$= \underline{b}$

~

At a Soln

$\underline{z} = 0.$

So ① $A\underline{x} = \underline{b}, \; \underline{x} \geq 0.$

⇒ feasible pt. for the Orig. LP!

② It's Easy to get a BFP for

Use cols of I!

$\Rightarrow \underline{z} = \underline{b}.$

@ a soln of

we use the BFP to start the Simplex for

$$\min \; c^T x \quad \text{st} \quad A\underline{x} = \underline{b}, \quad \underline{x} \geq 0.$$

$$\min \; c^T x \quad \text{st.} \quad -5x = -2, \quad x \geq 0.$$

⇓

$$\min \; \underline{e}^T \underline{z} \quad \text{s.t.} \quad \begin{bmatrix} -5 & 1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = -$$

st. $x, z \geq 0.$

Basis = ⇒ $z = -2$

☹

Try #2

$\min \underline{e}^T \underline{z}$

s.t. $[-5 \;\; -1]\begin{bmatrix} x \\ z \end{bmatrix} = -2$

$x, z \ge 0$

Basis

$\Rightarrow z = 2.$

2

In general

$\min e^T z$

s.t. $A\underline{x} + E\underline{z} = b$

$\underline{x}, \underline{z} \ge 0$

$E_{ii} = \begin{cases} +1 & b_i \ge 0 \\ -1 & b_i \le 0 \end{cases}$ $E_{ij} = 0$ $i \ne j$

This is called the Phase I or Crash LP.

$\Rightarrow$ you get a BFS.

Not always needed

$\min \underline{c}^T \underline{x}$

s.t. $A\underline{x} \le \underline{b}$

$\underline{x} \ge 0.$

$\Rightarrow \min c^T x$

s.t. $Ax + \underline{s} \le b$

$\underline{x}, \underline{s} \ge 0.$

End of LPs!