Lecture 20

Interior Pt methods

↳ LPs + Non-linear Eqs.

⊛ Brief intro to Newton's method for non-lin eq.

- Newton step for IP meth
- Centered vs. Affine (uncentered) step
- Starting pts.

Interior Point Methods for LPs

$$\min \ \underline{c}^T \underline{x} \quad \text{s.t.} \quad A\underline{x} = \underline{b}, \quad \underline{x} \geq 0$$

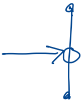
↓ Think about

$$\min \ \underline{c}^T \underline{x} - \tau \sum \log(x_i) \quad \text{s.t.} \quad A\underline{x} = \underline{b}$$

↓ this gives a sequence

$x(\tau), \lambda(\tau), s(\tau)$ that

Converge to a KKT pt as $\tau \to 0$.



Our goal is to use an alg to find

$\underline{x}_k, \underline{\lambda}_k, s_k$

St

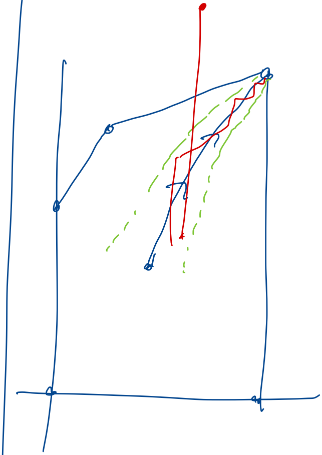
$\underline{x}_k, \underline{\lambda}_k, \underline{s}_k \to$

$\underline{x}^*, \underline{\lambda}^*, \underline{s}^*$

and the idea is that they should be tied to

$\boxed{x(\tau), \lambda(\tau), s(\tau)}$

somehow.



IP method try to Solve the KKT System directly!

$$\underline{c} - A^T \underline{\lambda} - \underline{s} = 0$$

$$A\underline{x} = \underline{b}$$

$$\underline{x}, \underline{s} \geq 0$$

$$\underline{x}^T \underline{s} = 0$$

If $x > 0$, +ve.

We can get a nonlinear System.

$A : m \times n$, $x : n \times 1$

$s : n \times 1$

$$0 = F(\underline{x}, \underline{\lambda}, \underline{s}) =$$

$$\begin{array}{ll} \begin{bmatrix} A^T \lambda + \underline{s} - c \\ A\underline{x} - \underline{b} \\ XS\underline{e} \end{bmatrix} & \begin{array}{l} \to n \text{ Eq} \\ \to m \text{ Eq.} \\ \to n \text{ Eq.} \\ x_1 s_1 \\ \vdots \\ \to x_i s_i \\ \vdots \\ x_n s_n \end{array} \end{array}$$

$$= \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$n + n + m$ variables

$n + m + n$ Eqs.

Basic IP Alg:

Solve $F(\underline{x}, \underline{\lambda}, \underline{s}) = 0$

(also, $x \geq 0$, $\underline{s} \geq 0$)

Idea: Use Newton's method to solve these Eq!

$\underline{x}_k, \underline{s}_k, \underline{\lambda}_k$ all come from Newton's method.

Aside

Goal: find $\underline{x}$ s.t.

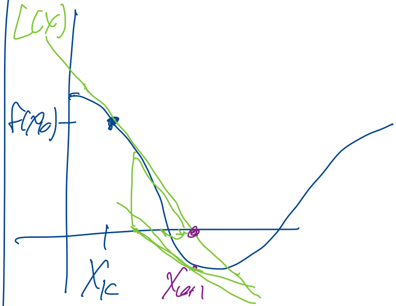
$F(\underline{x}) = 0$.

$F: \mathbb{R}^n \to \mathbb{R}^n$

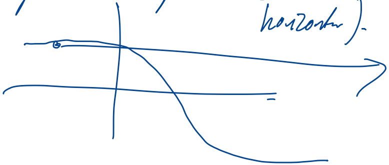
Idea:

- linearize F, Solve for 0.

In 1d:



④ Keypoint! A linear approx always has a unique 0 (unless horizontal).



Quiz

Given $f: \mathbb{R} \to \mathbb{R}$.
What is the eq. for the linear approx. of f at x_k?

i.e. what is $L(\underline{x})$?

- in 1d only.

Soln

$$f(\underline{x}) = f(\underline{x}_k) + f'(x_k)(x - x_k) + \frac{1}{2} f''(x_k)(x-x_k)^2 + \dots$$

$$f(\underline{x}) \approx f(\underline{x}_k) + f'(x_k)(x-x_k)$$

$$= L(x).$$

Solution

$$L(x) = 0$$

$$f(x_k) + f'(x_k)(x - x_k) = 0$$

$$\Rightarrow x = -\frac{f(x_k)}{f'(x_k)} + x_k.$$

Alg (1d):

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Computing

$$\frac{1}{\sqrt{x}}.$$

This comes up a lot when you want to normalize a vector!

In $\mathbb{R}^n$ what is Newton?

Same thing:

linearize + solve

$F(\underline{x}) : \mathbb{R}^n \to \mathbb{R}^n$

$$F(\underline{x}) = F(\underline{x}_k) + J(\underline{x}_k)(\underline{x} - \underline{x}_k) + \text{HOT}$$

(higher order terms)

$$\text{Approx. } L(\underline{x}) = F(\underline{x}_k) + J(\underline{x}_k)(\underline{x} - \underline{x}_k) = 0$$

$$x = \underline{x}_k - \overbrace{J(x_k)^{-1}(F(\underline{x}_k))}^{\text{solve } n \times n \text{ system of linear eq.}}$$

and iterate!

Newton vs.

Optimization

$x_{k+1} = x_k + \alpha_k p_k$

$p_k = -H_k^{-1} g_k$

- Relationship:

Opt $\Leftarrow$

Newton

non-lin Eq.

$$x_{k+1} = x_k - J_k^{-1} F_k$$

is Newton for

$$g(x) = 0$$

for non-lin Eq.

Back to IP methods

$$F(\underline{x}, \underline{\lambda}, \underline{s})$$

$$= \begin{array}{c} F_1 \\ F_2 \\ F_3 \end{array} \begin{bmatrix} A^T \lambda + \underline{s} - c \\ A\underline{x} - \underline{b} \\ X S \underline{e} \end{bmatrix} = [x_i s_i]$$

$$J = \begin{bmatrix} \partial F_1/\partial x & \partial F_1/\partial \lambda & \partial F_1/\partial s \\ \partial F_2/\partial x & \partial F_2/\partial \lambda & \partial F_2/\partial s \\ \partial F_3/\partial x & \partial F_3/\partial \lambda & \partial F_3/\partial s \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix}$$

$J(\underline{x}, \underline{\lambda}, \underline{s})$ is indep. of λ.

Basic IP method

Given $\underline{x}_0, \underline{\lambda}_0, \underline{s}_0$

Solve $$J(x_k, \lambda_k, s_k) \begin{bmatrix} \underline{p}_x \\ \underline{p}_\lambda \\ \underline{p}_s \end{bmatrix} = -F(x_k)$$

find $$\begin{bmatrix} x_{k+1} \\ \lambda_{k+1} \\ s_{k+1} \end{bmatrix} = x + \sigma \begin{bmatrix} p_x \\ p_\lambda \\ p_s \end{bmatrix}$$ s.t.

$$x_{k+1}, s_{k+1} \geq 0.$$

An important note!

We usually solve

$$F(x_{\text{new}}) = \begin{bmatrix} 0 \\ 0 \\ \sigma \tau \underline{e} \end{bmatrix}$$

Centering step!

Station pts

$$\min_{\underline{x}} \; \frac{1}{2} \underline{x}^T \underline{x} \quad \text{s.t.} \quad A\underline{x} = \underline{b}$$

$$\min_{\underline{s}, \lambda} \; \frac{1}{2} \underline{s}^T \underline{s} \quad \text{s.t.} \quad A^T \underline{\lambda} + \underline{s} = c$$

$$A\underline{x} = \underline{b}, \checkmark \quad A^T \lambda + \underline{s} = c$$

But soln $x_i s_i < 0$.