Lecture 7 HW3 posted last night.

An algorithm for CLS - Constrained Least Squares

$\min \|Ax-b\|_2^2$ s.t. $Cx=d$

- null space via QR
- null space via SVD
- how to find the initial point
- * Sports ranking.

Next, linearly constrained optimality conditions: min f(x) s.t. Cx = d

1. Null space (like CLS)
2. Next class, all this via Lagrange multipliers

Next class online, no class here :(

TAs office hours moved Monday 10-11am

Last time

$\min \|Ax-b\|_2^2$
s.t. $Cx=d$

and we used
x_c s.t. $Cx_c=d$
and null$(C) \Leftarrow\Rightarrow Z$ is a basis

to get the unconstrained problem

$\min_y \|A(x_c+Zy)-b\|_2^2$
s.t. nothing.

How to get Z
C: $p \times n$
$QR = C^T$
$n \times p$

$R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ p rows, n-p rows

R_1 $p\times p$ square (invertible)

$\Rightarrow$ partition Q's cols.
$Q = [\,Q_1 \;\; Q_2\,]$ (p, n-p) s.t.

$QR = Q_1R_1 + Q_2 \cdot 0 = C^T$

$\rightarrow$ w/ a little algebra you can show
Q_2 is a basis for null(C).
Why?
$CQ_2 = 0$
and it's of the right dimension.

$\Rightarrow$ ALG for Z.
Compute QR of C^T, use last n-p cols of Q.

Alg 2: Use the SVD.

$C = U\Sigma V^T$

U: $p\times p$ ortho.

Σ: $p \times n$ $\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_p & \end{bmatrix}$

diag. $\sigma_i > 0$

$V = n\times n$ ortho.

V^T, $\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$ p, n-p

$C \cdot V_2 = 0$
$\Leftarrow\Rightarrow$
$C\cdot V = U\Sigma$
$C\cdot [V_1\, V_2] = U\Sigma$

So V_2 is a basis for null(C) as well.

How to get x_c!
Use QR of C^T to do this!

We want
$Cx_c = d$
any x_c will do.
Recall $C^T = QR$ ($n\times n$, $n \times p$)

$Cx_c = R^T \underbrace{Q^T x_c}_{w} = d$

$R^T w = d$

$\rightarrow \begin{bmatrix} R_1^T & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = d$

$\Leftarrow\Rightarrow R_1^T w_1 = d$
$\Rightarrow w_1 = (R_1^T)^{-1} d$

$x_c = Q_1 (R_1^T)^{-1} d$ (n, p, p)

$Q = [Q_1\; Q_2]$ like before!.

As an ALG for CLS
Input: A, b, C, d

1. Compute QR of C^T.
   $\rightarrow$ gets Q_1, Q_2, R_1 solve sys.
2. Compute $x_c = Q_1 \cdot (R_1^T \backslash d)$
3. Solve
   $\min \|b - A(x_c + Q_2 y)\|_2^2$
   for y.
4. Return $x_c + Q_2 y$.

Least Sq. Sports Ranking.

Data from games:

team i plays team j.
p_i vs p_j
points, goals etc. — points, goals etc.

Goal: Compute a team strength s_i st.

$s_i - s_j \approx p_i - p_j$.

$\min_{s} \sum_{\text{games } (i,j)} ((s_i - s_j) - (p_i - p_j))^2$

$\sum s_i = 0$. or
$\sum s_i = 1$.

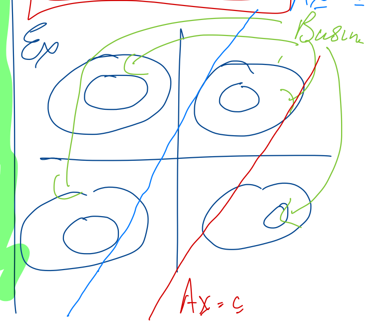
$\Downarrow$

$\min \|Ms - d\|_2^2$
st. $e^T s = \alpha$.

The idea in general
$\min f(x)$
st. $\cancel{Cx=d}$ $Ax = b$

!! Note $A \rightarrow C$ here! !!

Ex



Approach: Mirror LS.

1. find x_A st $Ax_A = b$
2. find Z a basis for null of A.
3. Solve
   $\min_y f(x_A + Zy)$ ①
   Unconstrained.

Op

Opt conditions for ①

Necessary
"$g(y)$" $= 0$
"$H(y)$" $\succeq 0$

Sufficient
"$g(y)$" $= 0$.
"$H(y)$" $\succ 0$

But this uses g, H from f(y) Not f(x).

"$f(y)$" $= f(x(y))$
$= f(x_A + Zy)$
$x(y) = x_A + Zy$

$\dfrac{\partial f}{\partial y} =$ "$g(y)$"

Use chain rule!

"$\dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial y}$"

But this fails in terms of dims :(

Right way to get it:

$\dfrac{\partial f}{\partial y} = \left(\dfrac{\partial x}{\partial y}\right)^T \dfrac{\partial f}{\partial x}$

$= Z^T g(x) = 0.$

"$H(y)$" $= Z^T H(x) Z \succeq 0$.

$\min f(x)$
s.t. $Ax = b$

Let Z be a basis for null (A).
Then

Necessary
$Z^T g(x) = 0$
$Z^T H(x) Z \succeq 0$
$Ax = b$

Sufficient
$Z^T g(x) = 0$
$Z^T H(x) Z \succ 0$
$Ax = b$
$\Rightarrow$ x is a local min

Terminology

$Z^T g(x)$ is called Reduced / projected gradient.

$Z^T H(x) Z$ is Reduced / projected Hessian.

Quiz

$\min (x_1-1)^2 + 3(x_2+1)^2 - 8x_1$
st. $x_1 = -x_2$
$\Leftarrow\Rightarrow x_1 + x_2 = 0 \Leftarrow\Rightarrow \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$QR(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) \Rightarrow \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$

$Z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Notice:
$x_2 = -x_1 \Rightarrow$
$\min (x_1-1)^2 + 3(-x_1+1)^2 - 8x_1$
$\Rightarrow$
$\min 4x_1^2 - 16x_1 + 4$
$\Rightarrow x_1 = 2$
$x_2 = -2$

$g(x) = \begin{bmatrix} 2(x_1-1)-8 \\ 6(x_2+1) \end{bmatrix}$

$H(x) = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$

$Z^T H(x) Z = 8$

$Z^T g(x) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 2(x_1-1)-8 \\ 6(x_2+1) \end{bmatrix} = 0$

1 eq. for x_1, x_2
Using $Ax = b$ gives 1 more eq.