Leakage-Resilience of Shamir's Secret Sharing: Identifying Secure Evaluation Places

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Abstract

Side-channel attacks are significant threats to secret sharing schemes; even a single-bit probe per share partially reveals an additively shared secret. Shamir's secret-sharing is a more promising alternative – its security hinges on where the secret-sharing polynomial is evaluated. Although most evaluation places yield secure schemes, identifying them is critical.

We initiate research into such classifier constructions over prime fields. This work considers schemes where shares of all parties are needed for reconstruction and, in this context, presents:

1. An algorithm to classify evaluation places as secure or vulnerable against parity leakage.
2. Modulus choices where the classifier above extends to arbitrary single-bit probes into each share.

On the way, it discovers new bit-probing attacks on Shamir's scheme.

We introduce new techniques to analyze the security of secret-sharing schemes against side-channel threats, connecting their leakage resilience to the orthogonality of square wave functions; the orthogonality, in turn, depends on the 2-adic valuations of rational approximations. These techniques are new to the security analysis of secret sharings and possibly of broader interest.

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1 Introduction

Beginning with the seminal works of Kocher et al. [34, 35, 12], ingenious *side-channel attacks* have repeatedly revealed cryptographic secrets. For example, *secret-sharing schemes* keep their secrets safe when only a few shares are compromised; side-channel attacks accumulate partial information from all shares to infer sensitive information. Securing our secret-sharing schemes against various side-channel threats has become even more compelling due to the ongoing NIST standardization efforts [10], where they are widely used in key distribution and other higher-level primitives like secure computation. *Local leakage resilience* [4, 24] is a security metric for secret sharings against a broad spectrum of side-channel threats that leak from each share independently. Local leakages are surprisingly powerful; even single-bit probes into every share partially reveal an additively shared secret [41, 1, 43, 19].

Shamir's secret sharing [56] is a more promising alternative – its security depends on where the secret-sharing polynomial is evaluated. Most evaluation places, for instance, ensure that the cumulative leakage from bit probes into shares is statistically independent of the secret [41, 45]. One may naïvely attempt to instantiate a locally leakage-resilient Shamir's scheme by choosing evaluation places at random. However, this strategy is *susceptible to adversarially set randomness*, where vulnerable evaluation places may be picked unbeknownst to the honest parties. Towards neutralizing this threat, a natural question arises:

Question: *Is there an algorithm to determine whether the picked evaluation places yield a locally leakage-resilient Shamir's secret sharing?*

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Any meaningful classifier in this context must have the following features.

1. *No false positives.* No evaluation places can be incorrectly determined to be leakage-resilient; otherwise, adversarially set randomness can ensure they are picked.
2. *A small number of false negatives.* Ideally, the algorithm should correctly identify most (or at least a significant fraction) of the leakage-resilient evaluation places.
3. *Efficiency.* The runtime of the classifier should not be "prohibitively large."

We initiate the research into constructing such classifiers for Shamir's secret-sharing over *prime fields.*

Summary of our results. We consider Shamir's schemes where *shares of all parties are required to reconstruct the secret* and investigate their security against *arbitrary single bit-probe in each share.* We present such classifiers for *Mersenne* and *Fermat* prime modulus. Our algorithms have $\mathsf{poly}(\log p)$ running time and $\sqrt{p}\cdot\mathsf{poly}(\log p)$ false negatives for prime modulus p. The technical workhorse is our classifier for the specific leakage that obtains each share's parity; this classifier works for *arbitrary prime modulus.* We present *new bit-probing attacks* to demonstrate the insecurity of evaluation places.

Summary of our key technical challenge. For an arbitrary prime modulus $p \geqslant 3$, define the parity function $\mathrm{LSB}\colon F_p \to F_2$ by $\mathrm{LSB}(x) := 0$, for $x \in \{0, 2, \dots, (p-1)\}$; otherwise, $\mathrm{LSB}(x) := 1$. Fix arbitrary elements $\alpha_1, \alpha_2 \in F_p$.

Technical Question: *For a uniformly random* $X \in F_p$,
are the distributions $\mathrm{LSB}(\alpha_1 \cdot X)$ *and* $\mathrm{LSB}(\alpha_2 \cdot X)$ *statistically independent?*

Answering this technical question is challenging because $x \mapsto \mathrm{LSB}(x)$ is a *non-linear map.*[1]

The answer. We prove that *the distributions are independent if (and only if)* $\alpha_1 \cdot \alpha_2^{-1} = u \cdot v^{-1}$ *such that* $u, v \in \left\{-\left\lceil\sqrt{p}\right\rceil, \dots, 0, \dots, \left\lceil\sqrt{p}\right\rceil\right\}$ *and the 2-adic valuation of* u/v *is* $\neq 0$.[2] Correlated distributions translate into *new side-channel attacks.*

Maji et al. [45] recently constructed such classifiers over *characteristic-2 composite-order fields.* The analogous map in their technical analysis is F_2-linear, making their technical question approachable via elementary "rank arguments." Analyzing non-linear leakage is technically challenging; in the related literature on repairing Reed-Solomon codewords, *non-linear repairing* was only recently addressed [15, 14]. Appendix A summarizes this discussion and other prior relevant works motivating this research.

Due to the ubiquity of the additive and Shamir's schemes, it is crucial to determine their security; this work contributes to this effort. Although easy to analyze, new constructions cannot replace these popular ones in security technologies. Section 6 presents natural follow-up questions and hurdles in approaching them.

[1] Answering this technical question is equivalent to estimating the following exponential sum, where $\{x\}$ represents the fractional part of x and $\mathrm{i} := \sqrt{-1}$.

$$\sum_{X=0}^{p-1} \exp\Big(\,\mathrm{i}\cdot\pi p\cdot(\,\{\alpha_1 X/p\} + \{\alpha_2 X/p\}\,)\,\Big).$$

[2] Our characterization finds an appropriate *rational approximation* u/v. The *2-adic valuation* of u/v is the highest power of 2 in u minus the highest power of 2 in v. So, the distributions are independent for relatively prime u, v when u or v is even. For relatively prime odd u, v, the dependence between these two distributions is $1/|u\cdot v|$; the distributions are nearly independent when $|u\cdot v|$ is large.

2 Basic Definitions & Our Formal Problem Statement

Shamir's secret sharing. Shamir's secret-sharing scheme among n parties with reconstruction threshold k over a finite field F and distinct evaluation places $\alpha_1, \alpha_2, \dots, \alpha_n \in F^*$ proceeds as follows. To share a secret $s \in F$, sample a random F-polynomial $P(X)$ such that $\deg P < k$ and $P(0) = s$. Define the shares: $s_1 := P(\alpha_1)$, $s_2 := P(\alpha_2)$, $\dots$, and $s_n := P(\alpha_n)$. Denote this secret-sharing by $\mathsf{ShamirSS}(n, k, \vec{\alpha})$ and the joint distribution of the shares by $\mathsf{Share}(s)$ – other parameters will be clear from the context. *This work only considers $n = k$.*

Representing prime field elements. Consider a prime field F_p of order p, where $2^{\lambda-1} < p < 2^\lambda$ and λ is the security parameter. The elements of F_p are represented as λ-bit binary strings representing the elements $\{0, 1, \dots, (p-1)\}$.

▶ Remark 1. For a Fermat prime $p = 2^\lambda + 1$, elements of F_p require $(\lambda + 1)$ bits in their binary representation. However, only the binary representation of 2^λ has 1 in the most significant bit. For simplicity of presentation, we assume that elements are represented using λ bits only; disregarding the element $2^\lambda \in F_p$ adds only an additive $1/p$ slack to the analysis.

Leakage functions & families. This work studies *physical bit leakage* $\mathrm{PHYS}_i \colon F_p \to \{0, 1\}$ that outputs the i-th least significant bit, where $i \in \{0, 1, \dots, \lambda - 1\}$. For example, PHYS_0 (also referred to as LSB) outputs 0 for the elements in $\{0, 2, \dots, (p-1)\}$, where $p \geqslant 3$, and PHYS_1 outputs 0 for the elements in $\{0, 1, 4, 5, \dots\}$. For $\mathbf{i} = (i_1, i_2, \dots, i_n) \in \{0, 1, \dots, \lambda - 1\}^n$, the *leakage function* $\vec{\mathrm{PHYS}}_{\mathbf{i}} \colon F^n \to \{0, 1\}^n$ leaks the i_t-th bit of the t-th share, where $t \in \{1, 2, \dots, n\}$. For a secret $s \in F$, the joint distribution of the leakage is $\vec{\mathrm{PHYS}}_{\mathbf{i}}(\mathsf{Share}(s))$. We consider two *leakage families*.

1. Physical bit leakage family: $\mathrm{PHYS} := \left\{ \vec{\mathrm{PHYS}}_{\mathbf{i}} \ :\ \mathbf{i} = (i_1, \dots, i_n) \in \{0, 1, \dots, \lambda - 1\}^n \right\}$.
2. LSB leakage family: $\mathrm{LSB} := \left\{ \vec{\mathrm{PHYS}}_{\mathbf{0}} \right\}$, where $\mathbf{0} = (0, 0, \dots, 0)$

Insecurity & randomized construction. *Insecurity* of $\mathsf{ShamirSS}(n, k, \vec{\alpha})$ against a leakage family $\mathcal{F}$ is:

$$\varepsilon_{\mathcal{F}}(\vec{\alpha}) := \max_{f \in \mathcal{F}} \ \max_{s \in F^*} \ \mathsf{SD}(\ f(\mathsf{Share}(0))\ ,\ f(\mathsf{Share}(s))\ ). \tag{1}$$

Low insecurity implies the statistical independence of the leakage from the secret, i.e., the *secret-sharing is locally leakage-resilient* [4, 24].

High insecurity indicates a leakage function can distinguish the secret 0 and some $s^* \in F^*$ using the leakage. Maji et al. [41] analyzed the insecurity against the PHYS leakage family when *evaluation places were chosen randomly.* Their result implies the following corollary for prime modulus $p \geqslant 3$ and $n = k \geqslant 2$.

For randomly chosen evaluation places $\vec{\alpha} \in \left(F_p^*\right)^n$, the insecurity $\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant p^{-1/2}$ with probability $\geqslant 1 - p^{-1/2}$.

Recently, [45] extended the randomized construction from prime fields to composite ones.

Our work investigates the security against the leakage family PHYS; i.e., the adversary obtains arbitrary *one physical bit leakage from each share.*

Our Research Question: *Given evaluation places $\vec{\alpha}$ and prime modulus p, identify whether (1) $\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \ \leqslant\ p^{-1/2}$ or (2) $\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \ >\ p^{-1/2}$.*

If $\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) > p^{-1/2}$, then output a secret $s^* \in F_p^*$ such that the shares of 0 can be distinguished from the shares of s^* with (roughly) $\varepsilon_{\mathrm{PHYS}}(\vec{\alpha})$ advantage. *All algorithms must be computationally efficient – runtime is a polynomial in λ.* Furthermore, concrete security analysis (over asymptotic analysis) is prioritized.

3 Our Results

Below, for $x, y, z \in \mathbb{R}$, the expression $x = y \pm z$ is a concise representation for "$x \in [y-z, y+z]$." For example, "x is close to y" is expressed using $x = y \pm \varepsilon$, for a small ε. Section 5 presents a high-level overview of the critical technical ideas underlying our results.

Technical Result: Security against LSB Leakage when $n = 2$. Consider *arbitrary prime $p \geqslant 3$* (not just a Mersenne/Fermat prime) and the LSB leakage. The technical workhorse for our results is the classifier for $(n, k) = (2, 2)$; other results bootstrap from it.

1. Figure 1 presents our efficient algorithm to classify $\vec{\alpha}$ as secure or not. If our algorithm classifies $\vec{\alpha}$ as secure, then

$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \leqslant \frac{1+8^{5/4}}{\sqrt{p}} \quad + \frac{13/2}{p} \leqslant \frac{14.46}{\sqrt{p}},$$

   which is exponentially small in the security parameter λ. The number of false negatives is $\mathcal{O}\left(\sqrt{p} \cdot \log p\right)$.
2. We present an efficient adversary that generates $s^* \in F^*$ such that it distinguishes the secret 0 from s^* by leaking the LSB of each share with an advantage

$$\geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \quad - \frac{2 \cdot 8^{5/4}}{\sqrt{p}} - \frac{13}{p} \geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) - \frac{26.91}{\sqrt{p}}.$$

   Therefore, our efficient leakage attack achieves a comparable distinguishing advantage when the insecurity $\varepsilon_{\mathrm{LSB}}(\vec{\alpha})$ is significant.

Section 7 presents the corollary statements relevant to LSB leakage and their proofs (Section 7.2 to Section 7.4 state and prove Corollary 11 to Corollary 13, respectively).

Result A: Security against Physical Bit Leakage when $n = 2$. For the $n = k = 2$ case, we analyze a prime field F_p, where p is a Mersenne/Fermat prime – primes of the form $2^\lambda \pm 1$. We reduce arbitrary physical bit leakage to LSB leakage for related evaluation places over these fields. In this context, our work proves the following results.

1. Figure 2 presents our efficient classifier against PHYS leakage. For $\vec{\alpha}$ classified secure, the insecurity is $\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant 14.46/\sqrt{p}$. The number of false negatives is $\mathcal{O}\left(\sqrt{p} \cdot (\log p)^2\right)$.
2. We present an efficient adversary that generates $(s^*, f) \in F_p^* \times \mathrm{PHYS}$ such that it distinguishes the secret 0 from secret s^* by leaking $f \in \mathrm{PHYS}$ from the shares with an advantage $\geqslant \varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \geqslant \varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) - 26.91/\sqrt{p}$. *These are new side-channel attacks; their existence demonstrates the tightness of our analysis.*
3. We explicitly identify secure evaluation places against PHYS leakage: all (α_1, α_2) satisfying $\alpha_2 \cdot \alpha_1^{-1} = 2^{\lfloor \lambda/2 \rfloor} - 1$. For these evaluation places $\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant 8.49/\sqrt{p}$.

Section 8 and Appendix E present the corollary statements relevant to these results and their proofs.

Result B: Security against Physical Bit Leakage when $n > 2$. Consider a Mersenne/Fermat prime field F_p, such that $p = 2^\lambda \pm 1$. Corollary 20 in Section 9 presents an efficient classifier for $\vec{\alpha}$ such that the corresponding $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ is secure to physical bit probes; the insecurity is at most $1/\sqrt{p}$. Section 9 then presents Theorem 19, which implies this corollary.

Appendix F.2 proves this theorem using Fourier analysis. Appendix F.1 presents the proof of Corollary 20.

Given evaluation places $\vec{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_n)$ we construct $\vec{\beta} := (\beta_1, \beta_2, \dots, \beta_n)$ using an efficiently computable linear map. We prove that if $\mathsf{ShamirSS}(2, 2, (\beta_1, \beta_2))$ has ε-insecurity against physical bit leakage, then $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ has 2ε-insecurity against physical bit leakage. Clarifications below highlight the subtlety of this classifier:

▶ Remark 2 (Clarifications). **1.** High insecurity of $\mathsf{ShamirSS}(2, 2, (\beta_1, \beta_2))$ *does not imply* high insecurity of $\mathsf{ShamirSS}(n, n, \vec{\alpha})$; *our result lifts security only in one direction.*

2. Can the security of $\mathsf{ShamirSS}(2, 2, (\alpha_i, \alpha_j))$, for all $1 \leqslant i < j \leqslant n$, imply the security of $\mathsf{ShamirSS}(n, n, \vec{\alpha})$? *This natural classifier has false positives.* Consider $n = 3$, a prime $p = 4w^2 + 6w + 9$, and evaluation places $\vec{\alpha} = (1, \sigma, \sigma^2)$, where $w \geqslant 4$, $w \neq 0 \mod 3$, and $\sigma = 2w \cdot 3^{-1}$. For example, $p = 97$ and $\sigma = 35$; Bunyakovsky conjecture [9] implies infinitude of such primes. Against LSB leakage, although every $\mathsf{ShamirSS}(2, 2, (\alpha_i, \alpha_j))$ is secure, $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ is $(2/\pi)^3 > 0.25$ insecure [43, 19]; Appendix G presents the details.

So, the following strategy suffices to construct secure schemes: (1) randomly sample $\vec{\alpha}$, (2) compute $\vec{\beta}$ using our linear map in Figure 3, and (3) test the security of $\mathsf{ShamirSS}(2, 2, (\beta_1, \beta_2))$ using Corollary 16.

4 Illustrating Technical Challenges: Resilience to LSB Leakage

Before presenting the technical overview of our results, it is instructive to illustrate the challenges of constructing such a classifier against the LSB leakage. Section 4.3 to Section 4.5 present three natural approaches that are incorrect. *How do we identify the evaluation places yielding* LSB *resilient sharing?* Their pattern is complex, our algorithm illustrates in Figure 1; Section 5.1 presents an overview. The exposition in Section 4.5 introduces several technical components of our algorithm.

4.1 A New Vulnerability to LSB Leakage

Consider Shamir's secret sharing scheme among $n = 2$ parties and threshold $k = 2$ over the prime fields F_p, where $p \geqslant 5$. To share the secret $s = 0$, choose a polynomial $P(X) := P_1 \cdot X$ for uniformly random $P_1 \in F_p$. Suppose the first share is $s_1 := P(1)$, the evaluation of $P(X)$ at $X = 1$, and the second is $s_2 := P(3)$. The two shares are elements of the following set:

$$(s_1, s_2) \in \Big\{ (P_1, 3 \cdot P_1) \, : \, P_1 \in F_p \Big\}. \tag{2}$$

The LSB attack – a specific bit probe – leaks each share's *parity*. For example, a share in the set $\{0, 2, 4, \dots, (p-1)\}$ has "even" parity, while a share in the complementary set $\{1, 3, \dots, (p-2)\}$ has "odd" parity. We aim to investigate the leakage joint distribution:

Is the LSB *leakage uniformly distributed over* $\{\mathit{even}, \mathit{odd}\} \times \{\mathit{even}, \mathit{odd}\}$*?*

Observation 1. Let us calculate the probability that the parity of s_1 differs from that of s_2. There are *two exhaustive cases.*

A: $\underline{\text{Share } s_1 = 2 \cdot x, \text{ where } x \in \mathbb{Z} \cap [p/6, 2p/6]}$: The parity of s_1 is even, and the parity of $s_2 = 3 \cdot s_1 = 6 \cdot x$ is odd (because of one " mod p" wraparound).

B: $\underline{\text{Share } s_1 = 2 \cdot x, \text{ where } x \in \mathbb{Z} \cap [4p/6, 5p/6]}$: The parity of s_1 is odd (because of one " mod p" wraparound), and $s_2 = 6 \cdot x$ is even (because of four " mod p" wraparounds).

Therefore, the probability of the parity of s_1 and s_2 being different is (roughly) 1/3; the leakage is *not* uniformly random.

Observation 2. Next, secret-share a uniformly random secret $s \in F_p$. The two shares are:

$$(s_1, s_2) = (s + P_1, s + 3 \cdot P_1), \tag{3}$$

where $s, P_1 \in F_p$ are uniformly and independently random. In this case, the leakage is uniformly random, and the probability of the parity of s_1 and s_2 being different is (roughly) 1/2 (because the two shares are also uniformly and independently distributed over F_p). By an averaging argument, there is a secret $s^* \in F_p$ such that the probability of the parity-of-shares-being-different is $\geqslant 1/2$. Our algorithms efficiently generate the secret s^*.

Conclusion. These two observations demonstrate that the LSB leakage is not independent of the secret; the secret 0 and s^* are distinguishable with advantage $\geqslant 1/2 - 1/3 = 1/6$. For brevity, we say that *the leakage is* 1/6*-dependent on the secret*, and this scheme is *vulnerable*.

This vulnerability extends to all $\alpha_1 \cdot \alpha_2^{-1} \in \{\pm 3, \pm 3^{-1}\}$ using properties of Generalized Reed Solomon codes [28]. Before our work, the only known vulnerable evaluation places satisfied $\alpha_1 \cdot \alpha_2^{-1} = -1$ [41, 1, 43, 19]. *Any* (α_1, α_2) *pair deemed insecure by our algorithm in Figure 1 will result in a new attack.*

4.2 An Example of Leakage-Resilience to LSB Leakage

Now consider evaluation places $(1, 2)$. In this case, the shares of 0 satisfy:

$$(s_1, s_2) \in \Big\{ (P_1, 2 \cdot P_1) \ : \ P_1 \in F_p \Big\}. \tag{4}$$

Similar to Observation 1, the probability of the parity of s_1 and s_2 being different is (roughly) 1/2; the leakage is uniformly random.[3] From this fact, using some additional analysis, one can prove that the LSB leakage is statistically independent of the secret.[4]

▶ Remark 3. Looking ahead, the evaluation places $(1, 2)$ is vulnerable to *physical bit leakage* over Mersenne/Fermat prime fields (see Remark 5).

4.3 Classifier Construction: First Attempt

Consider distinct evaluation places $\alpha_1, \alpha_2 \in F_p$. The corresponding shares of 0 are:

$$(s_1, s_2) \in \Big\{ (\alpha_1 \cdot P_1, \alpha_2 \cdot P_1) \ : \ P_1 \in F_p \Big\}. \tag{5}$$

Using the presentation above, it suffices to determine the probability of the two shares having different parity. This probability is computable in $\mathcal{O}(p)$ time by exhaustively considering all $P_1 \in F_p$. This brute-force algorithm is "efficient" only for small primes; however, any Shamir's scheme is $1/p$-dependent on the secret. Hence, Shamir's scheme is vulnerable to LSB leakage when the prime modulus is small. For large primes, as is standard in this line of research, the length of the binary representation of the elements in F_p, i.e., "$\lambda := \log_2 p$," denotes the problem size. *Any efficient algorithm must have a* $\mathsf{poly}(\lambda)$ *runtime, but this brute-force algorithm takes exponential time.*

[3] The shares have different parity when $(s_1, s_2) = (2 \cdot x, 4 \cdot x)$ and $x \in \mathbb{Z} \cap [p/4, 3p/4]$.

[4] The LSB leakage being uniformly random for secret 0 *does not* outrightly imply that the LSB leakage is independent of the secret. For example, it is possible that the probability of the shares having different parity is $> 1/2$ for half the secrets, and for the remaining secrets, this probability is $< 1/2$, such that their average is 1/2. However, our technical analysis rules out this possibility.

4.4 Classifier Construction: Second Attempt

Draw t elements $\left\{P_1^{(1)}, \ldots, P_1^{(t)}\right\}$ from F_p uniformly and independently at random. Compute the corresponding leakage for each sample and estimate the leakage distribution from these samples. Using the tightness of the Chernoff bound, the accuracy of this estimation is only $\mathsf{poly}(1/t)$, which is *too coarse-grained for any tractable number of samples.* So, this strategy will have false positives.

4.5 Classifier Construction: Third Attempt

This section focuses on developing an efficient algorithm. Let $(\alpha_1 = 1, \alpha_2)$ be the evaluation places. Extrapolating from the examples in Section 4.1 and Section 4.2, a "reasonable conjecture" would be the following characterization.

- If α_2 is odd: Declare "LSB leakage is $1/2\alpha_2$-dependent on the secret."
- (Else) If α_2 is even: Declare "LSB leakage is independent of the secret."

Fascinatingly, *this classifier also has false positives.* For example, consider a prime $p = 6t + 1$, where t is a large integer. Consider the evaluation places $(\alpha_1, \alpha_2) = (1, 2t + 2)$; the algorithm above misclassifies it as resilient to LSB leakage (because α_2 is even). However, these evaluation places are $1/30$-dependent on the secret.

Using properties of Generalized Reed-Solomon codes [28], the shares of 0 for evaluation places (α_1, α_2) and evaluation places (u, v) are identical when $\alpha_1 \cdot \alpha_2^{-1} = u \cdot v^{-1}$; represented by $(u, v) \in [\alpha_1 : \alpha_2]$. In our example, for instance, $(3, 5) \in [1 : 2t + 2]$ and, as in the previous examples, the shares are $(6x, 10x)$, for $x \in F_p$. The two shares have different parity when

$$x \in \mathbb{Z} \cap \left(\left[\frac{3p}{30}, \frac{5p}{30}\right] \cup \left[\frac{6p}{30}, \frac{9p}{30}\right] \cup \left[\frac{10p}{30}, \frac{12p}{30}\right] \cup \left[\frac{18p}{30}, \frac{20p}{30}\right] \cup \left[\frac{21p}{30}, \frac{24p}{30}\right] \cup \left[\frac{25p}{30}, \frac{27p}{30}\right] \right).$$

The probability of this event is $1/2 - 1/30$; therefore, the LSB leakage is $1/30$-dependent on the secret.

5 Technical Overview

5.1 Technical Result: LSB Leakage $(n, k) = (2, 2)$

We outline our classification algorithm for $(n, k) = (2, 2)$ and, en route, highlight our technical contributions (Figure 1 presents the pseudocode).

Step 1. The prime modulus p and the distinct non-zero evaluation places $\alpha_1, \alpha_2 \in F_p$ are inputs to the LSB classification algorithm. The security/vulnerability of evaluation places (α_1, α_2) is identical to any evaluation places (u, v) satisfying $\alpha_1 \cdot \alpha_2^{-1} = u \cdot v^{-1}$ (follows from Generalized Reed-Solomon codes' properties [28]). We find "small norm" $u, v \in \left\{-\lceil\sqrt{p}\rceil, \ldots, 0, 1, \ldots, \lceil\sqrt{p}\rceil\right\}$ with the property mentioned above – a Dirichlet approximation problem. We solve it with a small constant multiplicative slack using the LLL [39] algorithm in $\mathsf{poly}(\lambda)$ runtime, where $\lambda := \lceil \log_2(p + 1)\rceil$ (Appendix B has the details). The reasoning for choosing "small norm" u, v will be evident in Step 3.

Step 2. We aim to compute the probability that the shares s_1 and s_2 have different parity when the secret is 0; represent this probability by $1/2 + \varepsilon$. If ε has a small magnitude, the leakage is independent of the secret; otherwise, the LSB leakage can distinguish two secrets with an advantage of $|\varepsilon|$. By exhaustively considering all possible shares, one can compute ε in $\mathcal{O}(p)$ time, which is, unfortunately, exponential in our problem size λ.

Input. Distinct evaluation places $\alpha_1, \alpha_2 \in F^*$

Output. Decide whether the evaluation places (α_1, α_2) are secure to the LSB leakage attack

Algorithm.

1. Define the equivalence class

$$[\alpha_1 : \alpha_2] := \left\{ (u, v) \colon u = \Lambda \cdot \alpha_1, v = \Lambda \cdot \alpha_2, \Lambda \in F^* \right\}.$$

Use the LLL [38] algorithm to (efficiently) find $(u, v) \in [\alpha_1 : \alpha_2]$ such that

$$u, v \in \{-B, -(B-1), \dots, 0, 1, \dots, (B-1), B\} \mod p,$$

where $B := \left\lceil 2^{3/4} \cdot \sqrt{p} \right\rceil$. Refer to Figure 4 in Appendix B.

2. Remark: Our algorithm interprets $u, v \in \{-B, \dots, 0, 1, \dots, B\} \mod p$ as integers below.
3. Compute $g = \gcd(u, v)$.
4. If $u \cdot v / g^2$ is even: Declare $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ is secure against LSB leakage
5. (Else) If $u \cdot v / g^2$ is odd and $\left| u \cdot v / g^2 \right| \geqslant \sqrt{p}$: Declare $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ is secure against LSB leakage
6. (Else) Declare that the security of $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ against LSB attacks may be insecure

Figure 1 Identify secure evaluation places for Shamir's secret sharing against LSB leakage.

Step 3. To develop an efficient algorithm to compute ε, we express the quantity ε as the inner product of two oscillatory $\{\pm 1\}$ sequences, approximated by the following integral.

$$\int_0^1 \operatorname{sign} \sin(2\pi |u| \cdot t) \cdot \operatorname{sign} \sin(2\pi |v| \cdot t) \ \mathrm{d}t.$$

Here, $\operatorname{sign} \sin(2\pi |u| \cdot t) \in \{\pm 1\}$ is a *square wave* that oscillates $|u|$ times in the domain $[0, 1]$. The integral above measures the similarity/dependence between the two square waves, the first oscillating $|u|$ times and the second oscillating $|v|$ times. The error of our approximation is directly proportional to the total number of oscillations of the square waves. The approximation error is $\leqslant (|u| + |v|)/p = \mathcal{O}\left(1/\sqrt{p}\right)$, exponentially small in λ, for small norm u, v. For simplicity, the presentation below ignores this approximation error.

Step 4. Finally, we compute ε by obtaining a closed-form expression for the integral. For $g := \gcd(|u|, |v|)$ and $\rho := |u| \cdot |v| / g^2$, we prove that:

$$\varepsilon = \begin{cases} 0, & \text{if } \rho \text{ is even} \\ 1/2\rho, & \text{if } \rho \text{ is odd.} \end{cases}$$

When $\varepsilon = 0$, Shamir's scheme with evaluation places (α_1, α_2) is resilient to LSB leakage. Otherwise, there is an efficient algorithm to distinguish secret 0 from some $s^* \in F_p$ using the LSB leakage. Our toy example in Section 4.1 is insecure because $\rho = 1/6$ (since $u = \alpha_1 = 1$ and $v = \alpha_2 = 3$). One way for evaluation places α_1, α_2 to be secure is: there are small-norm u, v such that $\alpha_1 \cdot \alpha_2^{-1} = u \cdot v^{-1}$ and the highest powers of 2 dividing u and v are different (in which case ρ is even). Our classification algorithm for arbitrary physical bit probes will build on the result outlined in this section.

▶ Remark 4. Our work connects the security of secret-sharing schemes against leakage attacks with the orthogonality properties of a family of square waves; see, for example, [59, 30, 29]. Various families of square waves, like the ones by Haar [27], Walsh [60], and Rademacher [51], are central to science and engineering. These techniques are new to the security analysis of secret sharings and possibly of broader interest.

5.2 Overview of Result A: Physical Bit Leakage $(n, k) = (2, 2)$

Suppose the evaluation places are $\vec{\alpha} = (\alpha_1, \alpha_2)$. We aim to *determine whether Shamir's secret-sharing scheme with these evaluation places is secure against all physical bit leakage attacks in Mersenne prime fields.* For $i, j \in \{0, 1, \ldots, \lambda - 1\}$, consider the physical bit leakage attack $\overrightarrow{\mathrm{PHYS}}_{i,j}$. This leakage attack leaks the i-th LSB of the share s_1 and the j-th LSB of the share s_2. For a Mersenne prime p and an element $x \in F_p$, the binary representation of $x \cdot 2^{-1}$ is the *right rotation* of the binary representation of x by one position. Therefore, $\overrightarrow{\mathrm{PHYS}}_{i,j}$ leakage with evaluation places (α_1, α_2) is identical to the LSB leakage with evaluation places $(2^{-i} \cdot \alpha_1, 2^{-j} \cdot \alpha_2)$. By Generalized Reed-Solomon codes' properties [28], the leakage is identical to the LSB leakage with evaluation places $(2^{j-i} \cdot \alpha_1, \alpha_2)$. Consequently,

$$\varepsilon_{\mathrm{PHYS}}(\ (\alpha_1, \alpha_2)\ ) = \max\big\{\ \varepsilon_{\mathrm{LSB}}(\ (\alpha_1, \alpha_2)\ ),\ \varepsilon_{\mathrm{PHYS}}(\ (2\alpha_1, \alpha_2)\ ),\ \ldots,\ \varepsilon_{\mathrm{PHYS}}(\ (2^{\lambda-1}\alpha_1, \alpha_2)\ )\ \big\}.$$

Thus, security against PHYS leakage reduces to a sequence of LSB security estimations. Figure 2 presents this pseudocode.

▶ Remark 5 (An Edge Case). The algorithm determining the security of Shamir's secret-sharing scheme to LSB attack requires the evaluation places to be distinct. Even though α_1 and α_2 are distinct, it may be the case that $2^t\alpha_1 = \alpha_2$, for some $t \in \{0, 1, \ldots, \lambda - 1\}$. So, the call to the "LSB security check subroutine" with the argument $(2^t\alpha_1, \alpha_2)$ would be invalid. Lemma 15 proves that this edge case is insecure. This case captures why evaluation places $(1, 2)$ are insecure against physical bit leakage.

```
Input. Distinct evaluation places $\alpha_1, \alpha_2 \in F_p^*$, and $p$ is a Mersenne/Fermat prime.
Output. Decide whether the evaluation places $(\alpha_1, \alpha_2)$ are secure to an arbitrary single physical bit leakage per share.
Algorithm.
1. If there is $t \in \{0, 1, \ldots, \lambda - 1\}$ such that $2^t\alpha_1 = \alpha_2$: Return insecure
2. For $t \in \{0, 1, \ldots, \lambda - 1\}$:
    a. Call the algorithm in Figure 1 with evaluation places $(2^t\alpha_1, \alpha_2)$
    b. If the algorithm returns "may be insecure," return may be insecure
3. Declare $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ is secure against physical bit attacks.
```

Figure 2 Identify secure evaluation places for Shamir's secret sharing against physical bit leakage.

For *Fermat prime* p, $x \in F_p$, and $i \in \{1, 2, \ldots, \lambda - 1\}$ we prove following identity.[5]

$$\mathrm{PHYS}_{i-1}(x) = \mathrm{PHYS}_i(2x + 1). \tag{6}$$

Therefore, $\mathrm{PHYS}_i(x) = \mathrm{LSB}(2^{-i} \cdot x + 2^{-i} - 1)$. Like the Mersenne prime case above, arbitrary physical bit leakage translates into LSB leakage, except the map here is an *affine map* instead

[5] For primes other than Mersenne and Fermat primes, there is no such affine transformation.

of a *linear map*. As a result, the secret $s^* \in F_p^*$ witnessing the maximum insecurity is different; it is still efficiently computable. See Section 8.1 for details.

5.3 Overview of Result B: Physical Bit Leakage $n = k > 2$

Our objective is to choose n distinct evaluation places $\alpha_1, \alpha_2, \ldots, \alpha_n \in F^*$ such that the corresponding $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ is secure against physical bit leakage attacks. We prove a lifting theorem (Theorem 19) that proves the following result. Given evaluation places $\vec{\alpha}$, consider $\vec{\beta}$ related to Lagrange multipliers (where $i \in \{1, 2, \ldots, n\}$):

$$\beta_i := \left(\alpha_i \prod_{j \neq i} (\alpha_i - \alpha_j)\right)^{-1}.$$

Now consider the $\mathsf{ShamirSS}(2, 2, (\beta_i, \beta_j))$ secret-sharing scheme for all distinct $i, j \in \{1, 2, \ldots, n\}$. Suppose one of these secret-sharing schemes is secure against physical bit leakage. In that case, the $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ secret-sharing scheme is also secure. More concretely, if the insecurity of $\mathsf{ShamirSS}(2, 2, (\beta_i, \beta_j))$ is (at most) ε, for some distinct $i, j \in \{1, 2, \ldots, n\}$, then the $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ secret-sharing scheme is (at most) 2ε insecure.

Whether the evaluation places of $\mathsf{ShamirSS}(2, 2, (\beta_i, \beta_j))$ is secure can be determined efficiently using our algorithm in Figure 2. We can use this algorithm to detect whether our chosen $\vec{\alpha}$ has such a secure (β_i, β_j) pair of evaluation places. Corollary 20 formally states this result; its proof is entirely Fourier-analytic, and Appendix F presents it.

Input. Distinct evaluation places $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (F_p^*)^n$, and p is a Mersenne or Fermat prime

Output. Decide whether the evaluation places $\vec{\alpha}$ are secure to all physical bit leakage attacks

Algorithm.

1. For $i \in \{1, 2, \ldots, n\}$, compute

$$\beta_i := \left(\alpha_i \prod_{j \neq i} (\alpha_i - \alpha_j)\right)^{-1}.$$

2. If there exist $1 \leqslant i < j \leqslant n$ such that $\mathsf{ShamirSS}(2, 2, (\beta_i, \beta_j))$ is secure per the algorithm in Figure 2, then declare that $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ is secure.
3. Otherwise, the algorithm states that $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ may be insecure.

Figure 3 Identify secure evaluation places for $\mathsf{ShamirSS}(n, n)$ against physical bit leakage.

▶ Remark 6. One may hope that picking a random $\vec{\alpha}$ yields a random $\vec{\beta}$. *However, this naïve conjecture is false.* Some $\vec{\beta}$ are generated with higher-than-average probability, while some are never generated. We prove that (β_1, β_2) are (nearly) independent and uniform for $n \geqslant 3$.

6 Future Research Directions

There are several natural research directions for future work. A few immediate ones and their respective technical hurdles are presented below.

LSB **classifier construction for $n > 2$.** To illustrate the challenges, consider $n = 3$ and evaluation places $(\alpha_1, \alpha_2, \alpha_3)$. The rational approximation problem will require finding small-norm u, v, w such that $\alpha_1 : \alpha_2 : \alpha_3 = u : v : w$. Dirichlet approximation theorem only guarantees $|u|, |v|, |w| \leqslant p^{(n-1)/n}$. Therefore, *the accuracy error in estimating the summation by an integral will be* $p^{(n-1)/n}/p = p^{-1/n} \gg p^{-1/2}$, for $n \geqslant 3$.
Moreover, for $\varphi(x) = \operatorname{sign}\sin(2\pi x)$, *the estimate of the integral below is not known.*

$$\int_0^1 \varphi(ut) \cdot \varphi(vt) \cdot \varphi(wt)\,\mathrm{d}t. \tag{7}$$

Arbitrary physical bit leakage in general prime modulus. Against arbitrary physical bit leakage, extension to general prime modulus seems challenging. For example, when $n = 2$, the technical challenge is to characterize (α_1, α_2) such that the distributions $\mathrm{PHYS}_i(\alpha_1 X)$ is independent of $\mathrm{PHYS}_j(\alpha_2 X)$, where X is chosen uniformly at random. The bottleneck is to establish an integral that estimates this expression for a *general prime modulus.*

More physical probes. Consider $(n, k) = (2, 2)$, evaluation places (α_1, α_2), a Mersenne prime modulus p, and physical bit leakage probing the first share twice & the second share once. The technical problem is to show the independence of the following three distributions

$$\Big(\ \mathrm{PHYS}_i(\alpha_1 X),\ \mathrm{PHYS}_j(\alpha_1 X),\ \mathrm{PHYS}_k(\alpha_2 X)\ \Big),$$

where $X \in F_p$ is chosen uniformly at random. The analysis *reduces to estimating the integral in Equation 7*, where $u = 2^{-i}\alpha_1$, $v = 2^{-j}\alpha_1$, and $w = 2^{-k}\alpha_2$, which is not known.

More general (n, k). For concreteness, consider $(n, k) = (3, 2)$ and resilience to LSB leakage. This resilience requires t-wise independence of the leakage bits, where $k \leqslant t \leqslant n$. The 2-wise independence of leakage bits can be tested using the classifier in Figure 1. The 3-wise independence test has identical hurdles as the "LSB classifier construction for $n > 2$" case discussed above. There are evaluation places where *the* LSB *leakage is 2-wise independent but 3-wise correlated* for $(n, k) = (3, 2)$. The evaluation places of Appendix G with $(n, k) = (3, 2)$ (instead of $(n, k) = (3, 3)$) have this property.

7 Security against Least Significant Bit Leakage

This section presents our results regarding the security of Shamir's secret-sharing scheme when $n = k = 2$ against the LSB leakage. We begin with a powerful technical result.

▶ **Theorem 7** (Technical Result). *Consider the* $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ *secret-sharing scheme over a prime field F_p, where $p \geqslant 3$.*

$$\max_{s \in F} \mathsf{SD}\Big(\vec{\mathsf{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathsf{LSB}}(\mathsf{Share}(s))\Big)$$
$$= \begin{cases} \dfrac{4(|u| + |v|) - (3/2)}{p}, & \text{if } |u| \cdot |v|/g^2 \text{ is even,} \\[2ex] \cos^2(\pi/2p) \cdot \dfrac{g^2}{|u| \cdot |v|} \pm \dfrac{4(|u| + |v|) - (3/2)}{p} & \text{if } |u| \cdot |v|/g^2 \text{ is odd,} \end{cases}$$

where $\alpha_1 \cdot \alpha_2^{-1} = u \cdot v^{-1}$ *and* $g = \gcd(|u|, |v|)$.
Furthermore, for $s = \pm(u^{-1} \cdot v - 1)^{-1} \in F_p^*$, *if*

$$\mathsf{SD}\Big(\vec{\mathsf{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathsf{LSB}}(\mathsf{Share}(s))\Big) > \frac{4(|u| + |v|) - (3/2)}{p}$$

then there is an efficient distinguisher to distinguish the secret 0 and s with advantage at least

$$\cos^2(\pi/2p)\cdot\frac{g^2}{|u|\cdot|v|}\quad-\frac{4(|u|+|v|)-(3/2)}{p}$$

using the LSB *leakage on the secret shares.*

Essentially, this theorem helps estimate the insecurity efficiently. Section 7.1 presents the proof outline for this result and Appendix C presents the full proof. With this theorem, we will state and prove the corollaries mentioned in Section 3.

7.1 Proof outline of Theorem 7

For any $s\in F^*$, we start by obtaining a closed-form estimate of

$$\mathsf{SD}\Big(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\Big).$$

Then, we can solve for the optimal $s\in F^*$ that maximizes the statistical distance. Below, we present a high-level overview of the proof of Theorem 7.

Step 1. We connect the statistical distance between the leakages to the difference between two sums of oscillatory functions. We define the function $\mathrm{sign}_p\colon\mathbb{Z}\to\pm1$.

$$\mathrm{sign}_p(X):=\begin{cases}+1, & \text{if } X\in\{0,1,\dots,(p-1)/2\}\quad\bmod p\\ -1, & \text{if } X\in\{-(p-1)/2,\dots,-1\}\quad\bmod p.\end{cases}$$

For $k,\ell,\Delta\in F$, we define the following measurement of similarity between two lines kT and $\ell(T-\Delta)$ on F.

$$\Sigma^{(\Delta)}_{k,\ell}:=\sum_{T\in F}\mathrm{sign}_p(kT)\cdot\mathrm{sign}_p(\ell(T-\Delta)).\tag{8}$$

▶ **Lemma 8.** *Consider the* $\mathsf{ShamirSS}(2,2,(\alpha_1,\alpha_2))$ *secret-sharing scheme over a prime field* F_p. *For any secret* $s\in F_p$ *and* $(u,v)\in[\alpha_1\colon\alpha_2]$,

$$\mathsf{SD}\Big(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\Big)=\frac{1}{2p}\cdot\left|\Sigma^{(0)}_{u,v}-\Sigma^{(\Delta)}_{u,v}\right|,$$

where $\Delta:=\left(s\cdot2^{-1}\right)\cdot\left(u^{-1}-v^{-1}\right)$, *a linear automorphism over* F_p.

Appendix C.1 proves Lemma 8.

Step 2. Next, our objective is to estimate the sum $\frac{1}{p}\cdot\Sigma^{(\Delta)}_{k,\ell}$ using the integral $I^{(\delta)}_{k,\ell}$ defined as an inner product of two square wave functions as follow.

$$I^{(\delta)}_{k,\ell}:=\int_0^1\mathrm{sign}\sin(2\pi|k|\cdot t)\cdot\mathrm{sign}\sin(2\pi|\ell|\cdot(t-\delta))\ \mathrm{d}t.$$

▶ **Lemma 9.** *For any* $k,\ell,\Delta\in F_p$, *and* $\delta=\frac{\mathrm{sign}_p(\Delta)\cdot|\Delta|}{p}\in\mathbb{Q}$,

$$\frac{1}{p}\cdot\Sigma^{(\Delta)}_{k,\ell}=\mathrm{sign}_p(k)\cdot\mathrm{sign}_p(\ell)\cdot I^{(\delta)}_{|k|,|\ell|}+\frac{\mathrm{sign}_p(k\Delta)-\mathrm{sign}_p(\ell\Delta)}{p}\quad\pm\frac{4(|k|+|\ell|)-2}{p}.$$

Appendix C.2 proves Lemma 9.

Step 3. Finally, we compute the value of the integral $I^{(\delta)}_{k,\ell}$.

▶ **Lemma 10.** *For any $k, \ell \in \{1, 2, \dots\}, \delta \in \mathbb{R}$, and $g = \gcd(k, \ell)$*

$$I^{(\delta)}_{k,\ell} = \begin{cases} 0, & \text{if } k \cdot \ell / g^2 \text{ is even} \\ \cos(2\ell\pi \cdot \delta) \cdot \dfrac{g^2}{k\ell}, & \text{if } k \cdot \ell / g^2 \text{ is odd.} \end{cases}$$

Appendix C.3 proves Lemma 10. Intuitively, if the highest power of 2 dividing k is different from the highest power of 2 dividing ℓ, then $k\ell/g^2$ is even and $I^{(\delta)}_{k,\ell} = 0$. If the highest power of 2 dividing k is identical to the highest power of 2 dividing ℓ, then $k\ell/g^2$ is odd and $I^{(\delta)}_{k,\ell} \neq 0$.

Step 4. Sequentially performing the substitutions above, we can estimate the statistical distance using the integrals, which yields Theorem 7 after maximizing over every $s \in F^*$.

Efficient distinguisher construction. We present an efficient maximum likelihood distinguisher in Appendix C.7.

7.2 Insecurity Estimation: Statement and Proof of Corollary 11

Using Theorem 7, we prove that the estimated insecurity achieved by our classifier in Figure 1 is close to the true insecurity.

▶ **Corollary 11.** *Consider distinct evaluation places $\vec{\alpha} = (\alpha_1, \alpha_2)$ and the corresponding* ShamirSS$(2, 2, \vec{\alpha})$ *secret-sharing scheme over the prime field F_p, where $p \geqslant 3$. Let $(u, v) \in [\alpha_1 \colon \alpha_2]$ such that $|u|, |v| \leqslant B$, where $B = \lceil 8^{1/4} \cdot \sqrt{p} \rceil$. Let $g = \gcd(|u|, |v|)$. Define*

$$\varepsilon^{(\text{est})}_{\text{LSB}}(\vec{\alpha}) := \begin{cases} 0, & \text{if } |u| \cdot |v|/g^2 \text{ is even,} \\ \cos^2(\pi/2p) \cdot \dfrac{g^2}{|u| \cdot |v|}, & \text{if } |u| \cdot |v|/g^2 \text{ is odd.} \end{cases}$$

Then,

$$\varepsilon^{(\text{est})}_{\text{LSB}}(\vec{\alpha}) = \varepsilon_{\text{LSB}}(\vec{\alpha}) \quad \pm \left(\frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p} \right).$$

Proof. Use the LLL algorithm [39] to efficiently find $(u, v) \in [\alpha_1 \colon \alpha_2]$ with properties mentioned in the corollary (see Appendix B for details). Observe that the LHS of the expression in Theorem 7 is identical to $\varepsilon_{\text{LSB}}(\vec{\alpha})$ by our definition in Equation 1. From this observation, the corollary is immediate. ◀

Next, we state the corollaries mentioned in Section 3 through this tight estimation.

7.3 Insecurity Identification: Statement of Corollary 12

▶ **Corollary 12.** *Consider distinct evaluation places $\vec{\alpha} = (\alpha_1, \alpha_2)$ and the corresponding* ShamirSS$(2, 2, \vec{\alpha})$ *secret-sharing scheme over the prime field F_p, where $p \geqslant 3$. Suppose the algorithm in Figure 1 determines $\vec{\alpha}$ to be secure. Then,*

$$\varepsilon_{\text{LSB}}(\vec{\alpha}) \leqslant \frac{1 + 8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

Among all possible distinct evaluation places $\alpha_1, \alpha_2 \in F_p^*$, the algorithm of Figure 1 determines at least

$$\geqslant 1 \quad - \frac{\frac{1}{4} \cdot \sqrt{p} \cdot \ln p + \frac{3}{2} \cdot \sqrt{p} + \frac{1}{2}}{p - 2} \geqslant^{(*)} 1 \quad - \left(\frac{\ln p}{4\sqrt{p}} + \frac{5/2}{\sqrt{p}} \right).$$

fraction of them to be secure. The $(*)$ *inequality holds for any prime* $p \geqslant 11$*.*

Appendix C.4 proves Corollary 12.

7.4 Advantage of Adversary: Statement of Corollary 13

▶ **Corollary 13.** *Consider distinct evaluation places* $\vec{\alpha} = (\alpha_1, \alpha_2)$ *and the corresponding* $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ *secret-sharing scheme over the prime field* F_p*, where* $p \geqslant 3$*. If* $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) > \frac{2\cdot 8^{5/4}}{\sqrt{p}} + \frac{13}{p}$*, then there is an efficient algorithm that generates* $s \in F_p^*$ *and can distinguish the secret* 0 *from the secret* s *with an advantage*

$$\geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \quad - \frac{2\cdot 8^{5/4}}{\sqrt{p}} - \frac{13}{p}$$

by leaking the LSB *of the secret shares.*

Consider an efficient adversary outputs the s indicated in Theorem 7. After observing the leakage (ℓ_1, ℓ_2), the algorithm performs maximum likelihood decoding – computes whether secret 0 or secret s is more likely to have generated the observed leakage. Then, it predicts the most likely of the two events.

Appendix C.5 provides a full proof of the distinguishing advantage and security guarantee of this adversary.

8 Security against all Physical Bit Leakage

We consider $\mathsf{ShamirSS}(n = 2, k = 2, (\alpha_1, \alpha_2))$ over the prime field F_p of order $p \geqslant 3$. Let λ be the security parameter. This section considers Mersenne and Fermat primes, i.e., primes of the form $p = 2^\lambda \pm 1$. Some initial Mersenne primes are $3, 7, 31, 127, 8191$, and 131071, and Fermat primes are $3, 5, 17, 257$, and 65537.

Mersenne and Fermat's primes satisfy the following property.

▶ **Proposition 14.** *Let* λ *be the security parameter. Fix an arbitrary* $i \in \{0, 1, 2, \dots, \lambda - 1\}$*. For all* $x \in F_p$*,*

$$\mathrm{PHYS}_i(x) = \begin{cases} \mathrm{PHYS}_0(2^{-i}\cdot x) & \text{if } p = -1 \mod 2^{i+1} \\ \mathrm{PHYS}_0(2^{-i}\cdot x + (2^{-i}-1)) & \text{if } p = 1 \mod 2^{i+1} \end{cases}$$

We prove this proposition in Appendix E.1.

8.1 Leakage attack when $2^k\alpha_1 = \alpha_2$

Although $\alpha_1 \neq \alpha_2$, it may be possible that $2^k\alpha_1 = \alpha_2$, for some $k \in \{0, 1, \dots, \lambda - 1\}$. We prove that the secret-sharing scheme is insecure, taking care of this case in the algorithm of Figure 2. Suppose we are leaking the i-th bit of the first secret share and the j-th bit of the second secret share, such that $j - i = k$.

Suppose the secret is $s \in F$. Then, the secret share at evaluation place α is $s + u\alpha$, for uniformly random $u \in F$. The joint distribution of leakage is

$$(\mathrm{PHYS}_i(s + u\alpha_1), \mathrm{PHYS}_j(s + u\alpha_2)).$$

Let $v := u2^{-j}$ and $t := s2^{-j}$. Appendix E.2 uses Proposition 14 to show that when the order of the field is a Mersenne or Fermat's prime, the joint distribution of leakage is equivalent as (for uniformly random $v \in F$)

$$\big(\mathrm{PHYS}_0(t2^k + v\alpha_1 2^k), \mathrm{PHYS}_0(t + v\alpha_2)\big) \equiv \big(\mathrm{PHYS}_0(t2^k + v\alpha_2), \mathrm{PHYS}_0(t + v\alpha_2)\big),$$

because $2^k\alpha_1 = \alpha_2$. When $t = 0$, both the leakage bits are identical. On the other hand, for $t = t^* := (2^k - 1)^{-1}$, the joint distribution of leakage is

$$(\mathrm{PHYS}_0(1 + t^* + v\alpha_1), \mathrm{PHYS}_0(t^* + v\alpha_2))$$

These two leakage bits are different with $(1 - 1/p)$ probability. Therefore, one can distinguish the secret 0 and secret t^*2^j with $(1 - 1/p) \sim 1$ advantage by leaking $\vec{\mathrm{PHYS}}_{i,j}$; whence the following lemma follows. See Appendix E.2 for details.

▶ **Lemma 15.** *Let F be the prime field of order $p = 2^\lambda \pm 1$. Consider distinct evaluation places $\alpha_1, \alpha_2 \in F^*$ such that $2^k \cdot \alpha_1 = \alpha_2$ for some $k \in \{0, 1, \ldots, \lambda - 1\}$. Then,*

$$\mathsf{SD}\left(\vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(s))\right) \geqslant 1 - \frac{1}{p},$$

where $i, j \in \{0, 1, \ldots, \lambda - 1\}$, $j - i = k \mod \lambda$. If $p = 2^\lambda - 1$, $s = (2^k - 1)^{-1} \cdot 2^j$ and if $p = 2^\lambda + 1$, $s = (2^k - 1)^{-1} \cdot 2^j - 1$.

8.2 Upper Bound on insecurity

▶ **Corollary 16.** *Let F be the prime field of order $p = 2^\lambda \pm 1$. Consider distinct evaluation places $\vec{\alpha} = (\alpha_1, \alpha_2)$ and the corresponding $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ secret-sharing scheme over the prime field F. Suppose the algorithm in Figure 2 determines $\vec{\alpha}$ to be secure. Then,*

$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant \frac{1 + 8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

Among all possible distinct evaluation places $\alpha_1, \alpha_2 \in F^*$, the algorithm of Figure 1 determines at least

$$\geqslant 1 \quad - \frac{\ln p}{\ln 2} \cdot \frac{\frac{1}{4} \cdot \sqrt{p} \cdot \ln p + \frac{3}{2} \cdot \sqrt{p} + \frac{1}{2}}{p - 2} \geqslant^{(*)} 1 \quad - \frac{\ln p}{\ln 2} \cdot \left(\frac{\ln p}{4\sqrt{p}} + \frac{5/2}{\sqrt{p}}\right).$$

fraction of them to be secure. The $(*)$ inequality holds for all $p \geqslant 11$.

The proof of this corollary can be found in Appendix E.3.

8.3 Derandomization

We conclude this section by presenting a 'derandomization' result that is a direct consequence of Theorem 7.

▶ **Corollary 17.** *Let F be the prime field of order $p = 2^\lambda - 1$ where $\lambda > 3$. Define $t := \lfloor \lambda/2 \rfloor$. Consider $\vec{\alpha} = (\alpha_1, \alpha_2) \in [1\colon 2^t - 1]$ respectively. Then*

$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant \frac{4 \cdot \left(2^{\lfloor \lambda/2 \rfloor} + 2^{\lceil \lambda/2 \rceil}\right) - 6}{p}.$$

A similar result holds for Fermat primes as well. Note that if $p = 2^\lambda + 1$ is a prime, then $\lambda/2$ is an integer because λ must be a power of 2.

▶ **Corollary 18.** *Let F be the prime field of order $p = 2^\lambda + 1$. Define $t := \lambda/2$. Consider $\vec{\alpha} = (\alpha_1, \alpha_2) \in [1\colon 2^t - 1]$ respectively. Then*

$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant \frac{8 \cdot 2^{\lambda/2} - 3/2}{p}.$$

See Appendix E.4 for their proofs.

9 Extension to arbitrary Number of Parties

We extend our derandomization results to Shamir's secret-sharing scheme with the reconstruction threshold k equal to the number of parties $n \in \{2, 3, \dots\}$. We begin by stating the following general lifting theorem.

▶ **Theorem 19.** *Consider $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ over a prime field F. For every $i \in \{1, 2, \dots, n\}$, define*

$$\beta_i := \left(\alpha_i \prod_{j \neq i} (\alpha_i - \alpha_j)\right)^{-1}.$$

Suppose there are two indices $1 \leqslant i^* < j^* \leqslant n$ such that $\mathsf{ShamirSS}(2, 2, (\beta_{i^*}, \beta_{j^*}))$ has ε-insecurity against physical bit leakages. Then, $\mathsf{ShamirSS}(n, n, (\alpha_1, \alpha_2, \dots, \alpha_n))$ has at most 2ε-insecurity against physical bit leakages.

The proof of this theorem is Fourier-analytic and uses properties of the Generalized Reed-Solomon (GRS) codes. Corollary 20 is a consequence of the above theorem (Theorem 19).

▶ **Corollary 20.** *Let F_p be the prime field of order $p = 2^\lambda \pm 1$ and $n \in \{2, 3, \dots\}$. Consider distinct evaluation places $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and the corresponding $\mathsf{ShamirSS}(n, n, \vec{\alpha})$ secret-sharing scheme over the prime field F_p. Suppose the algorithm in Figure 3 determines $\vec{\alpha}$ to be secure. Then,*

$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant \frac{2 \cdot 8^{5/4}}{\sqrt{p}} + \frac{13}{p}.$$

Among all possible distinct evaluation places $\vec{\alpha} \in (F_p^*)^n$, the algorithm of Figure 3 determines at least

$$1 - \left(\frac{1}{4 \ln 2} \cdot \frac{(\ln p)^2 \sqrt{p}}{p - n} + \frac{5}{2 \ln 2} \cdot \frac{(\ln p) \sqrt{p}}{p - n}\right)$$

fraction of them to be secure.

We present the full proof of Theorem 19 and Corollary 20 in Appendix F.

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A Prior Related Works

The literature on leakage resilience and related areas is vast. It is challenging to cover all of them exhaustively; representative ones, most relevant to our work, are presented below.

A.1 Physical bit probing attacks

Probing wires and introducing random faults into them seem innocuous but lead to devastating attacks – the more straightforward the attack, the greater a security threat it poses. For example, Boneh et al. [8] showed the vulnerability of computing RSA signatures to random fault injection into memory. Ishai, Sahai, and Wagner [32] introduced the *bit probing model* to theoretically investigate real-world threats posed by an adversary that can probe a bounded number of memory locations. Bit probes on a share can also be used to estimate its Hamming weight, which leads to realistic threats like (1) algebraic side-channel attacks (beginning with the work of Renauld et al. [53]) and (2) recent attacks like Hertzbleed [61]. Recently, Faust et al. [19] provided a more comprehensive presentation. Maji et al. [41] introduced the "parity-of-parities" attack on the *additive secret-sharing* scheme. This attack leaks the LSB of each share, and the parity of the leaked bits is correlated with the secret's parity; this attack leads to $(2/\pi)^n \approx 0.63^n$ insecurity [1, 43, 19]. This simple attack matches the upper bound on the insecurity of the additive secret-sharing scheme against *arbitrary local leakage* proved in [4, 5]. Maji et al. [41] & Costes and Stam [16] independently observed that *Shamir's secret-sharing* scheme inherits this vulnerability if the modulus and the evaluation places are carelessly chosen. *Our work identifies more vulnerable evaluation places using the same* LSB *attack.*

A.2 Local leakage resilience

Benhamouda et al. [4] introduced leakage-resilient secret-sharing, which was implicit in the work of Goyal and Kumar [24]. Several works have *constructed new secret-sharing schemes* resilient to leakage attacks [6, 2, 57, 3, 37, 7, 20, 57, 21, 31, 13, 47, 11, 11]. There is a significant interest in characterizing the leakage-resilience of practical secret-sharing schemes, like the additive and Shamir's secret-sharing scheme. [41, 45] proved that, for reconstruction threshold $k = 2$ and an arbitrary number of parties n, choosing evaluation places at random yields a leakage-resilient Shamir secret-sharing scheme with a high probability against physical bit leakage. A sequence of works also determined the optimal leakage attack [41, 1, 43]. Other Monte Carlo constructions have also been proposed in [46, 42]. Faust et al. [19] present additional motivations for this research direction from practical motivations and a security analysis against the Hamming weight leakage (specialized to Mersenne primes).

Another flavor of results characterizes the leakage-resilience of Shamir's secret-sharing scheme for a large number of parties. For example, when $k \geqslant 0.69n$, Shamir's secret sharing (with any evaluation places) is leakage-resilient to (arbitrary) one-bit local leakage. Here, the insecurity is exponentially small in n [4, 5, 46, 44, 33]. Contrast this with our scenario, where the insecurity is exponentially small in the security parameter, independent of n. [49] proved that Shamir's secret-sharing scheme is insecure to local leakage when n/k is large.

A.3 Reed-Solomon Code Repair

Guruswami and Wooters [25, 26] considered the exact repair problem for Reed-Solomon codes – an antithetical objective to leakage resilience. They aim to repair the codeword by obtaining partial information from each block. Subsequently, a large body of work focused on repairing Reed-Solomon codes [17, 18, 22, 23, 50, 58, 62, 52, 63, 64, 15, 14]. Massey [48] connected linear secret-sharing schemes and linear codes. For example, Shamir's secret-sharing scheme is the Massey secret-sharing scheme corresponding to (punctured) Reed-Solomon codes. Repairing strategies for Reed-Solomon codes translate into techniques to reconstruct the secret in Shamir's secret-sharing scheme. However, there are crucial differences though. The literature on Reed-Solomon codes evaluates the secret polynomial on *all* finite field elements; they have $n = \mathsf{card}(F)$. For leakage resilience, typically, the secret polynomial is evaluated at $n \leqslant \mathsf{poly}(\log \mathsf{card}(F))$ evaluation places; [15, Section VI] highlights this distinction. Furthermore, they need to reconstruct the entire secret; however, the family of information that they obtain from each block is more general than simply bit probes. *Even in this literature, reconstruction using a small number of (arbitrary) bits per share and prime fields has been challenging due to non-linearity [15, 14].*

A.4 Square wave function families

Various families of square waves find wide applications in science and engineering. For example, consider the ones proposed by Haar [27], Walsh [60], and Rademacher [51]. In our work, we connect the leakage resilience of secret-sharing schemes with the properties of another family of square waves (see, for example, [59, 30, 29])

$$\left\{ \operatorname{sign}\sin(2\pi k \cdot x) \right\}_{k\in\mathbb{Z}}.$$

Previous works [59, 30] have studied the orthogonality of this family of waves. We investigate, more generally, the "similarity" among these waves and their offsets – functions of the form $\operatorname{sign}\sin(2\pi k \cdot (x - \delta))$, for $\delta \in \mathbb{R}$. In our context, zero similarity coincides with orthogonality; non-zero similarity represents correlation.

A.5 Simultaneous Diophantine Approximation

Solving simultaneous Diophantine approximation problems is a well-studied problem. This problem arises when choosing a "good basis" for a lattice. In our context, for an odd prime p, given distinct $\alpha_1, \alpha_2 \in \{1, 2, \dots, p-1\}$, our objective is to find $q \in \{1, 2, \dots, p-1\}$ such that $q\alpha_1 \mod p$ and $q\alpha_2 \mod p$ are either in the range $\{1, \dots, \sqrt{p}\}$ or $\{p - \sqrt{p}, \dots, p-1\}$. The integers $q\alpha_1 \mod p$ and $q\alpha_2 \mod p$, intuitively, have "small norm $\mod p$." We will use the classical LLL algorithm [39] to efficiently achieve this objective (see Appendix B).

The Dirichlet approximation theorem [54, 55] states that, for any $\alpha \in \mathbb{R}^d$ and any positive integer N, there is a denominator $1 \leqslant q \leqslant N^d$ such that

$$\max_{i\in\{1,2,\dots,d\}} \{q\alpha_i\} \leqslant \frac{1}{N}. \qquad (9)$$

Computing this solution is computationally challenging [38]. However, we can efficiently solve this problem by slightly weakening the upper bound on q. The seminal LLL algorithm [39], in particular, for $\alpha \in \mathbb{Q}^d$, finds $1 \leqslant q \leqslant 2^{d(d+1)/4} \cdot N^d$ satisfying Equation 9.

B Solving Simultaneous Diophantine Equations

Figure 4 presents our algorithm. In this section, the "LLL algorithm" refers to the algorithm with the following guarantees.

▶ **Theorem 21** (LLL [39, Proposition 1.39])**.** *There exists a polynomial-time algorithm that, given a positive integer d and rational numbers $r_1, r_2, \ldots, r_d, \varepsilon$ satisfying $0 < \varepsilon < 1$, finds integers $s_1, s_2, \ldots, s_d$, and t for which*

$$|s_i - t \cdot r_i| \leqslant \varepsilon,$$

for $1 \leqslant i \leqslant d$ and $1 \leqslant t \leqslant 2^{d(d+1)/4} \cdot \varepsilon^{-d}$.

```
Input. $\alpha_1, \alpha_2 \in F^*$, where $F$ is the prime field of order $p$
Output. Elements $u, v \in F^*$ such that $(u, v) \in [\alpha_1 : \alpha_2]$ and

    $u, v \in \{-B, -(B-1), \ldots, 0, 1, \ldots, (B-1), B\} \mod p,$

where $B := \lceil 2^{3/4} \cdot \sqrt{p} \rceil$.
Algorithm.
1. Interpret $\alpha_1, \alpha_2 \in \{0, 1, \ldots, p-1\}$ as positive integers
2. Define $d = 2$
3. Define $r_1 = \alpha_1/p \in \mathbb{Q}$ and $r_2 = \alpha_2/p \in \mathbb{Q}$
4. Define $\varepsilon = B/p \in \mathbb{Q}$
5. Use the LLL algorithm to find integers $s_1, s_2$, and $t$
6. Interpret $t$ as an element of $F$. Define $u = \alpha_1 \cdot t \in F$ and $v = \alpha_2 \cdot t \in F$
```

Figure 4 Our Algorithm to obtain (u, v) from (α_1, α_2) using the LLL-algorithm.

Let us proceed to analyze our algorithm of Figure 4. The parameter setting needs to ensure that $t \leqslant 2^{d(d+1)/4}\varepsilon^{-d} < p$. Recall that $\varepsilon = B/p$. Substituting this value and rearranging, one needs to ensure that $2^{d(d+1)/4} \cdot p^{d-1} < B^d$. Therefore we have chosen $B = \lceil 2^{(d+1)/4} p^{1-1/d} \rceil$. Consequently, one can interpret t as an F^* element.

By definition, $(u, v) \in [\alpha_1 : \alpha_2]$ because $u = t \cdot \alpha_1$ and $v = t \cdot \alpha_2$. Next, note that

$$|\alpha_1 \cdot t - s_1 \cdot p| \leqslant \varepsilon \cdot p = B, \text{ and } |\alpha_2 \cdot t - s_2 \cdot p| \leqslant \varepsilon \cdot p = B.$$

This argument completes the analysis that for every (α_1, α_2) how we obtain $(u, v) \in [\alpha_1 : \alpha_2]$ such that u and v are "small (positive/negative) numbers."

C Proof of Technical Lemmas

C.1 Proof of Lemma 8

Proof of Lemma 8. Consider the $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ secret-sharing scheme over a prime field F_p. Consider an arbitrary secret $s \in F_p$ and evaluation places $(u, v) \in [\alpha_1 \colon \alpha_2]$.

$$
\begin{aligned}
&\mathsf{2SD}\Big(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\Big) \\
&\quad = \sum_{\vec{\ell} \in \{0,1\}^2} \left| \Pr\Big[\vec{\mathrm{LSB}}(\mathsf{Share}(0)) = \vec{\ell}\Big] - \Pr\Big[\vec{\mathrm{LSB}}(\mathsf{Share}(s)) = \vec{\ell}\Big] \right| \\
&\quad = \sum_{\vec{\ell} \in \{0,1\}^2} \Bigg| \mathop{\mathbb{E}}_{X}\left[\mathbb{1}_{\mathrm{LSB}^{-1}(\ell_1)}(uX) \cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_2)}(vX)\right] \\
&\qquad\qquad\qquad - \mathop{\mathbb{E}}_{X}\left[\mathbb{1}_{\mathrm{LSB}^{-1}(\ell_1)}(uX+s) \cdot \mathbb{1}_{\mathrm{LSB}^{-1}(\ell_2)}(vX+s)\right] \Bigg| \qquad (10)
\end{aligned}
$$

▷ Claim 22. For $\ell \in \{0, 1\}$ and $X \in F_p$, we have

$$
\mathbb{1}_{\mathrm{LSB}^{-1}(\ell)}(X) = \frac{1}{2}\Big(1 + (-1)^{\ell} \cdot \mathrm{sign}_p(X \cdot 2^{-1})\Big).
$$

Applying Claim 22 to Equation 10, we get

$$
\begin{aligned}
&\mathsf{2SD}\Big(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\Big) \\
&= \sum_{\vec{\ell} \in \{0,1\}^2} \Bigg| \mathop{\mathbb{E}}_{X}\left[\left(\frac{1 + (-1)^{\ell_1}\,\mathrm{sign}_p(uX \cdot 2^{-1})}{2}\right) \cdot \left(\frac{1 + (-1)^{\ell_2}\,\mathrm{sign}_p(vX \cdot 2^{-1})}{2}\right)\right] \\
&\qquad - \mathop{\mathbb{E}}_{X}\left[\left(\frac{1 + (-1)^{\ell_1}\,\mathrm{sign}_p((uX+s) \cdot 2^{-1})}{2}\right) \cdot \left(\frac{1 + (-1)^{\ell_2}\,\mathrm{sign}_p((vX+s) \cdot 2^{-1})}{2}\right)\right] \Bigg| \\
&= \frac{1}{4} \cdot \sum_{\vec{\ell} \in \{0,1\}^2} \Bigg| \mathop{\mathbb{E}}_{X}\left[\mathrm{sign}_p(uX \cdot 2^{-1}) \cdot \mathrm{sign}_p(vX \cdot 2^{-1})\right] \\
&\qquad\qquad - \mathop{\mathbb{E}}_{X}\left[\mathrm{sign}_p((uX+s) \cdot 2^{-1}) \cdot \mathrm{sign}_p((vX+s) \cdot 2^{-1})\right] \Bigg| \\
&= \Bigg| \mathop{\mathbb{E}}_{X}\left[\mathrm{sign}_p(uX \cdot 2^{-1}) \cdot \mathrm{sign}_p(vX \cdot 2^{-1})\right] \\
&\qquad\qquad - \mathop{\mathbb{E}}_{X}\left[\mathrm{sign}_p((uX+s) \cdot 2^{-1}) \cdot \mathrm{sign}_p((vX+s) \cdot 2^{-1})\right] \Bigg| \\
&= \frac{1}{p} \cdot \Bigg| \sum_{X \in F_p} \mathrm{sign}_p(uX \cdot 2^{-1}) \cdot \mathrm{sign}_p(vX \cdot 2^{-1}) \\
&\qquad\qquad - \sum_{X \in F_p} \mathrm{sign}_p((uX+s) \cdot 2^{-1}) \cdot \mathrm{sign}_p((vX+s) \cdot 2^{-1}) \Bigg| \\
&= \frac{1}{p} \cdot \Bigg| \sum_{Y \in F_p} \mathrm{sign}_p(uY) \cdot \mathrm{sign}_p(vY) - \sum_{Z \in F_p} \mathrm{sign}_p(uZ) \cdot \mathrm{sign}_p(v(Z - s \cdot 2^{-1} \cdot (u^{-1} - v^{-1}))) \Bigg|
\end{aligned}
$$

The last step uses the renaming $X\cdot 2^{-1}\mapsto Y$ (an F automorphism) and $(X+s\cdot u^{-1})\cdot 2^{-1}\mapsto Z$ (an F-automorphism). Therefore,

$$\mathsf{SD}\Big(\mathsf{LSB}(\mathsf{Share}(0))\ ,\ \mathsf{LSB}(\mathsf{Share}(s))\Big)=\frac{\left|\Sigma^{(0)}_{u,v}-\Sigma^{(\Delta)}_{u,v}\right|}{2p},$$

where $\Delta:=\left(s\cdot 2^{-1}\right)\cdot\left(u^{-1}-v^{-1}\right)$, a linear automorphism over F_p.

◀

C.2 Proof of Lemma 9

Recall that $\mathrm{sign}_p(X=0)=+1$ and $\mathrm{sign}(x=0)=0$. Due to this mismatch, we defined an intermediate function satisfying $\widetilde{\mathrm{sign}}_p(X=0)=0$.

$$\widetilde{\mathrm{sign}}_p(X):=\begin{cases}+1, & \text{if } X\in\{1,\dots,(p-1)/2\}\mod p\\ 0, & \text{if } X=0\mod p\\ -1, & \text{if } X\in\{-(p-1)/2,\dots,-1\}\mod p.\end{cases}\tag{11}$$

Analogously, we define

$$\widetilde{\Sigma}^{(\Delta)}_{k,\ell}:=\sum_{T\in F}\widetilde{\mathrm{sign}}_p(kT)\cdot\widetilde{\mathrm{sign}}_p(\ell(T-\Delta)).$$

So, our next objective is to relate the quantities $\Sigma^{(\Delta)}_{k,\ell}$ with $\widetilde{\Sigma}^{(\Delta)}_{k,\ell}$.

▷ Claim 23. For any $k,\ell,\Delta\in F_p$,

$$\Sigma^{(\Delta)}_{k,\ell}=\widetilde{\Sigma}^{(\Delta)}_{k,\ell}\quad+\left(\sum_{T\in\{0,\Delta\}}\mathrm{sign}_p(kT)\cdot\mathrm{sign}_p(\ell(T-\Delta))\right).$$

Proof of Claim 23. This claim follows directly from the definition of $\Sigma^{(\Delta)}_{k,\ell}$, $\mathrm{sign}_p(X)$, $\widetilde{\Sigma}^{(\Delta)}_{k,\ell}$, and $\widetilde{\mathrm{sign}}_p(X)$, for $k,\ell,\Delta\in F_p$. The primary observation is that $\mathrm{sign}_p(X)=\widetilde{\mathrm{sign}}_p(X)$, for all $X\in F_p^*$, and $\widetilde{\mathrm{sign}}_p(X=0)=0$.

$$\begin{aligned}\Sigma^{(\Delta)}_{k,\ell}&=\sum_{T\in F}\mathrm{sign}_p(kT)\cdot\mathrm{sign}_p(\ell(T-\Delta))\\&=\left(\sum_{T\in F}\widetilde{\mathrm{sign}}_p(kT)\cdot\widetilde{\mathrm{sign}}_p(\ell(T-\Delta))\right)+\left(\sum_{T\in\{0,\Delta\}}\mathrm{sign}_p(kT)\cdot\mathrm{sign}_p(\ell(T-\Delta))\right)\\&=\widetilde{\Sigma}^{(\Delta)}_{k,\ell}+\left(\sum_{T\in\{0,\Delta\}}\mathrm{sign}_p(kT)\cdot\mathrm{sign}_p(\ell(T-\Delta))\right)\end{aligned}$$

◀

▷ Claim 24 (Transference Property). For all $k,\Delta\in F_p, X\in\mathbb{Z}, X=X'\mod p, x=X'/p\in\frac{1}{p}\cdot\mathbb{Z}$, and $\delta=\Delta/p\in\frac{1}{p}\cdot\mathbb{Z}$,

$$\widetilde{\mathrm{sign}}_p(k\cdot(X-\Delta))=\varphi(k\cdot(x-\delta)).$$

▷ Claim 25. For $k, \Delta \in F_p$ and $x \in \frac{1}{p} \cdot \mathbb{Z}$, define $\delta := \frac{\Delta}{p} \in \frac{1}{p} \cdot \mathbb{Z}$ and $\delta' := \frac{\mathrm{sign}_p(\Delta) \cdot |\Delta|_p}{p} \in \frac{1}{p} \cdot \mathbb{Z}$. Then,

$$\varphi(k \cdot (x - \delta)) = \varphi(k \cdot (x - \delta')).$$

Proof of Claim 25. Consider the following exhaustive case analysis.

- **Case 1:** $\Delta \in \{0, 1, \ldots, (p-1)/2\}$.
  In this scenario, $\mathrm{sign}_p(\Delta) = 1, |\Delta|_p = \Delta$ and $\delta = \delta'$. Then, $\varphi(k \cdot (x - \delta)) = \varphi(k \cdot (x - \delta'))$.
- **Case 2:** $\Delta \in \{(p+1)/2, (p+3)/2, \ldots, p-1\}$.
  In this scenario, $\mathrm{sign}_p(\Delta) = -1, |\Delta|_p = p - \Delta$ and $\delta' = \delta - 1$. Then,

$$\begin{aligned}\varphi(k \cdot (x - \delta')) &= \varphi(k \cdot (x - \delta + 1)) \\ &= \mathrm{sign}(\sin(2\pi k \cdot (x - \delta + 1))) \\ &= \mathrm{sign}(\sin(2\pi k \cdot (x - \delta) + 2\pi k)) \\ &= \mathrm{sign}(\sin(2\pi k \cdot (x - \delta))) \\ &= \varphi(k \cdot (x - \delta))\end{aligned}$$

whence the claim. ◀

▷ Claim 26. For $k \in F_p$ and $x, \delta \in \frac{1}{p} \cdot \mathbb{Z}$, the following holds.

$$\varphi(k \cdot (x - \delta)) = \mathrm{sign}_p(k) \cdot \varphi(|k|_p \cdot (x - \delta)).$$

Proof of Claim 26. Similar to the previous claim, we prove this via exhaustive case analysis.

- **Case 1:** $k \in \{0, 1, \ldots, (p-1)/2\}$.
  We can see $|k|_p = k$, $\mathrm{sign}_p(k) = 1$, and $\varphi(k \cdot (x - \delta)) = \mathrm{sign}_p(k) \cdot \varphi(|k|_p \cdot (x - \delta))$ holds by simply plugging in the values.
- **Case 2:** $k \in \{(p+1)/2, (p+3)/2, \ldots, p-1\}$.
  Substituting $|k|_p = p - k$ and $\mathrm{sign}_p(k) = -1$ into $\mathrm{sign}_p(k) \cdot \varphi(|k|_p \cdot (x - \delta))$ gives us

$$\begin{aligned}&\mathrm{sign}_p(k) \cdot \varphi(|k|_p \cdot (x - \delta)) \\ &= \mathrm{sign}(\sin(2\pi |k|_p \cdot (x - \delta))) && \text{(Because } |k|_p = p - k) \\ &= \mathrm{sign}_p(k) \cdot \mathrm{sign}(\sin(2\pi (p - k) \cdot (x - \delta))) \\ &= \mathrm{sign}_p(k) \cdot \mathrm{sign}(\sin(2\pi (px - p\delta) - 2\pi k \cdot (x - \delta))) \\ &&& \text{(Because if } x, \delta \in \tfrac{1}{p} \cdot \mathbb{Z} \text{ then } px, p\delta \in \mathbb{Z}) \\ &= \mathrm{sign}_p(k) \cdot \mathrm{sign}(\sin(-2\pi k \cdot (x - \delta))) \\ &= -\mathrm{sign}_p(k) \cdot \mathrm{sign}(\sin(2\pi k \cdot (x - \delta))) && \text{(Because } \mathrm{sign}_p(k) = -1) \\ &= \mathrm{sign}(\sin(2\pi k \cdot (x - \delta))) \\ &= \varphi(k \cdot (x - \delta))\end{aligned}$$

This proves the claim. ◀

Given the Transference Property (Claim 24), Claim 25 and Claim 26, we observe that for $k, \ell, \Delta \in F_p, T \in F, t = T/p \in \mathbb{Q}$ and $\delta = \frac{\mathrm{sign}_p(\Delta) \cdot |\Delta|_p}{p} \in \mathbb{Q}$,

$$\begin{aligned}\widetilde{\Sigma}_{k,\ell}^{(\Delta)} &= \sum_{T \in F} \widetilde{\mathrm{sign}}_p(kT) \cdot \widetilde{\mathrm{sign}}_p(\ell(T - \Delta)) \\ &= \sum_{t \in \{\frac{0}{p}, \frac{1}{p}, \ldots, \frac{p-1}{p}\}} \mathrm{sign}_p(k) \cdot \mathrm{sign}_p(\ell) \cdot \varphi(|k|_p t) \cdot \varphi(|\ell|_p (t - \delta))\end{aligned}$$

▶ **Definition 27** (Number of Oscillations). *A Boolean function $f\colon [0,1] \to \{\pm 1\}$ oscillates at $x \in [0,1)$ if $f(x) \neq \lim_{h \to 0^+} f(x+h)$. The* number of oscillations *is the cardinality of the following set.*

$$\left\{x\colon f(x) \neq \lim_{h \to 0^+} f(x+h)\right\}.$$

Since our functions are periodic with period 1, counting the number of oscillations in the interval $[0,1)$ in our context suffices.

By straightforward counting, one concludes the following.

▷ Claim 28 (Counting Number of Oscillations). For any $|k|_p, |\ell|_p \in \{1,\dots,(p-1)/2\}$,

1. $\varphi(|\ell|_p(x-\delta))$ oscillates $(2|\ell|_p - 1)$ times, if $\delta \in \frac{1}{2|\ell|_p} \cdot \mathbb{Z}$
2. $\varphi(|\ell|_p(x-\delta))$ oscillates $2|\ell|_p$ times, if $\delta \notin \frac{1}{2|\ell|_p} \cdot \mathbb{Z}$
3. $\varphi(|k|_p x) \cdot \varphi(|\ell|_p(x-\delta))$ oscillates $2(|k|_p + |\ell|_p) - 2$ times, if $\delta \in \frac{1}{2|k|_p} \cdot \mathbb{Z} \cap \frac{1}{2|\ell|_p} \cdot \mathbb{Z}$
4. $\varphi(|k|_p x) \cdot \varphi(|\ell|_p(x-\delta))$ oscillates $2(|k|_p + |\ell|_p) - 1$ times, if $\delta \notin \frac{1}{2|k|_p} \cdot \mathbb{Z} \cap \frac{1}{2|\ell|_p} \cdot \mathbb{Z}$

We prove the following general result that connects the sum of a Boolean function to its integral.

▷ Claim 29 (Sum and Integral Connection). Fix an integer $n \in \{1,2,\dots\}$. Let $f\colon [0,1] \to \{\pm 1\}$ be a Boolean function that oscillates H times in the range $[0,1]$. Then,

$$\frac{1}{n} \cdot \sum_{t \in \left\{\frac{0}{n}, \frac{1}{n}, \dots, \frac{n-1}{n}\right\}} f(t) \in \int_0^1 f(t)\,\mathrm{d}t \pm \frac{2H}{n}.$$

Proof of Claim 29. Consider an interval $[r, r+1/n)$, for $r \in \{0/n, 1/n, \dots, (n-1)/n\}$. If f does not oscillate in this interval, then f is constant in the interval, and we conclude

$$\frac{1}{n} \cdot f(t) = \int_r^{r+1/n} f(t)\,\mathrm{d}t.$$

If f oscillates at some point in this interval, then (due to f being Boolean) we conclude

$$\frac{1}{n} \cdot f(t) \in [-1/n, 1/n] \subseteq \int_r^{r+1/n} f(t)\,\mathrm{d}t \pm \frac{2}{n}.$$

Adding this over all $r \in \{0/n, 1/n, \dots, (n-1)/n\}$ proves the claim. ◀

Claim 23 is essentially a consequence of Claim 28 and Claim 29 when

$$f(t) = \mathrm{sign}_p(k) \cdot \mathrm{sign}_p(\ell) \cdot \varphi(|k|_p t) \cdot \varphi(|\ell|_p(t-\delta)).$$

We are now ready to prove Lemma 9.

Proof of Lemma 9. For any $k, \ell, \Delta \in F_p$ and $\delta = \frac{\mathrm{sign}_p(\Delta) \cdot |\Delta|_p}{p} \in \mathbb{Q}$.

$$\frac{1}{p} \cdot \widetilde{\Sigma}_{k,\ell}^{(\Delta)} = \begin{cases} \mathrm{sign}_p(k) \cdot \mathrm{sign}_p(\ell) \cdot I_{|k|_p,|\ell|_p}^{(\delta)} \pm \dfrac{4(|k|_p + |\ell|_p) - 4}{p} & \text{if } \delta \in \dfrac{1}{2|k|_p} \cdot \mathbb{Z} \cap \dfrac{1}{2|\ell|_p} \cdot \mathbb{Z} \\ \mathrm{sign}_p(k) \cdot \mathrm{sign}_p(\ell) \cdot I_{|k|_p,|\ell|_p}^{(\delta)} \pm \dfrac{4(|k|_p + |\ell|_p) - 2}{p} & \text{if } \delta \notin \dfrac{1}{2|k|_p} \cdot \mathbb{Z} \cap \dfrac{1}{2|\ell|_p} \cdot \mathbb{Z} \end{cases}$$

Combining the two cases, we get

$$\frac{1}{p}\cdot\widetilde{\Sigma}^{(\Delta)}_{k,\ell} = \mathrm{sign}_p(k)\cdot\mathrm{sign}_p(\ell)\cdot I^{(\delta)}_{|k|_p,|\ell|_p} \pm \frac{4(|k|_p+|\ell|_p)-2}{p}.$$

Applying Claim 23, we conclude the following

$$\frac{1}{p}\cdot\Sigma^{(\Delta)}_{k,\ell} = \mathrm{sign}_p(k)\cdot\mathrm{sign}_p(\ell)\cdot I^{(\delta)}_{|k|,|\ell|} + \frac{\mathrm{sign}_p(k\Delta)-\mathrm{sign}_p(\ell\Delta)}{p} \quad \pm\frac{4(|k|+|\ell|)-2}{p}.$$

which completes the proof of Lemma 9. ◀

C.3 Proof of Lemma 10

To begin with, we formalize the orthogonal properties of the sine and cosine functions.

▶ **Proposition 30** (Orthogonality of Sine/Cosine Waves [36, Page 38]). *For $k,\ell\in\{1,2,\dots\}$*

$$\int_0^1 \sin(2k\pi t)\cdot\sin(2\ell\pi t)\,\mathrm{d}t = \begin{cases}0, & \text{if } k\neq\ell\\ 1/2, & \text{if } k=\ell.\end{cases}$$

$$\int_0^1 \sin(2k\pi t)\cdot\cos(2\ell\pi t)\,\mathrm{d}t = 0.$$

For the periodic *square wave* [59, 30, 29] $\varphi\colon\mathbb{R}\to\{-1,0,+1\}$.

$$\varphi(x) := \mathrm{sign}\sin(2\pi x),$$

[30] uses (basic) Fourier analysis and Proposition 30 to determine the Fourier expansion of $\varphi(x)$.

$$\varphi(x) = \sum_{\text{odd } n>0}\frac{4}{\pi n}\cdot\sin(2n\pi x). \tag{12}$$

We prove the following claim for standardization.

▷ Claim 31. For $k,\ell\in F_p$ and $\delta\in\mathbb{R}$, the following identity holds

$$I^{(\delta)}_{k,\ell} = I^{(\delta)}_{k/g,\ell/g}$$

where $g=\gcd(k,\ell)$.

Proof of Claim 31. Define $\psi^{(\delta)}_{k,\ell}(x) := \varphi(kx)\cdot\varphi(\ell\cdot(x-\delta))$.

Observe that $\psi^{(\delta)}_{k,\ell}(x)=\psi^{(\delta)}_{k,\ell}(x{+}1/d)$, for any d that divides both k and ℓ. Let $g=\gcd(k,\ell)$. So, from our observation, we conclude that $\psi^{(\delta)}_{k,\ell}$ has period $1/g$. Therefore, we conclude that

$$I^{(\delta)}_{k,\ell} = g\cdot\int_0^{1/g}\psi^{(\delta)}_{k,\ell}(t)\,\mathrm{d}t.$$

Next, note that $\psi^{(\delta)}_{k,\ell}(x)=\psi^{(\delta)}_{k/d,\ell/d}(d\cdot x)$, for any d that divides both k and ℓ. So we have

$$I^{(\delta)}_{k,\ell} = g\cdot\int_0^{1/g}\psi^{(\delta)}_{k/g,\ell/g}(gt)\,\mathrm{d}t.$$

By substituting the variable $r=gt$, we get

$$I^{(\delta)}_{k,\ell} = g\cdot\int_0^{1}\psi^{(\delta)}_{k/g,\ell/g}(r)\cdot\frac{1}{g}\cdot\mathrm{d}r = I^{(\delta)}_{k/g,\ell/g}.$$

◀

Previously only $I_{k,\ell}^{(0)}$ was studied [59, 30]. In particular, motivated by our application scenario, we study $I_{k,\ell}^{(\delta)}$, for all $\delta \in \mathbb{R}$. To begin our analysis, we assume that k and ℓ are relatively prime.

▷ Claim 32. For relatively prime $k, \ell \in F_p$ such that $k \cdot \ell$ is even, $I_{k,\ell}^{(\delta)} = 0$, for all $\delta \in \mathbb{R}$.

Proof of Claim 32. Suppose k is even and ℓ is odd. In this case, for any odd $m, n > 0$,

$$\begin{aligned}
&\sin\left(2n\pi \cdot k\left(\frac{1}{2}+t\right)\right) \cdot \sin\left(2m\pi \cdot \ell\left(\frac{1}{2}+t-\delta\right)\right) \\
&\quad = \sin\begin{pmatrix} 2n\pi \cdot kt \\ + \\ \underbrace{nk}_{\text{even}} \cdot \pi \end{pmatrix} \cdot \sin\begin{pmatrix} 2m\pi \cdot \ell(t-\delta) \\ + \\ \underbrace{m\ell}_{\text{odd}} \cdot \pi \end{pmatrix} \\
&\quad = \sin(2n\pi \cdot kt) \cdot \big(\, -\sin(2m\pi \cdot \ell(t-\delta))\ \big)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^1 \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t \\
&= \int_0^{1/2} \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t + \int_{1/2}^{1} \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t \\
&= \int_0^{1/2} \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t - \int_0^{1/2} \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t \\
&= 0. \qquad (13)
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_{k,\ell}^{(\delta)} &= \int_0^1 \varphi(kt) \cdot \varphi(\ell(t-\delta))\,\mathrm{d}t \\
&= \frac{16}{\pi^2} \sum_{\text{odd } n>0} \sum_{\text{odd } m>0} \frac{1}{mn} \int_0^1 \sin(2n\pi \cdot kt) \cdot \sin(2m\pi \cdot \ell(t-\delta))\,\mathrm{d}t \quad \text{(By Equation 12)} \\
&= 0 \quad \text{(By Equation 13)}
\end{aligned}$$

Lastly, if k is odd and ℓ is even, then

$$\begin{aligned}
&\sin\left(2n\pi \cdot k\left(\frac{1}{2}+t\right)\right) \cdot \sin\left(2m\pi \cdot \ell\left(\frac{1}{2}+t-\delta\right)\right) \\
&\quad = \sin\begin{pmatrix} 2n\pi \cdot kt \\ + \\ \underbrace{nk}_{\text{odd}} \cdot \pi \end{pmatrix} \cdot \sin\begin{pmatrix} 2m\pi \cdot \ell(t-\delta) \\ + \\ \underbrace{m\ell}_{\text{even}} \cdot \pi \end{pmatrix} \\
&\quad = \big(\, -\sin(2n\pi \cdot kt)\ \big) \cdot \sin(2m\pi \cdot \ell(t-\delta))
\end{aligned}$$

which is the same as the previous case. Therefore, Equation 13 again holds and the proof of this case still goes through; the final result remains the same. ◀

▷ Claim 33. For relatively prime $k, \ell \in \{1, 2, \dots\}$ such that $k \cdot \ell$ is odd,

$$I_{k,\ell}^{(\delta)} = \frac{\cos(2\ell\pi \cdot \delta)}{k\ell},$$

for all $\delta \in \mathbb{R}$. This also shows that $I_{k,\ell}^{(\delta)}$ achieves its maximum at $\delta \in \frac{1}{\ell} \cdot \mathbb{Z}$, and the minimum at $\delta \in \frac{1}{2\ell} + \frac{1}{\ell} \cdot \mathbb{Z}$.

To prove this claim, we first need to generalize Proposition 30 a bit.

▷ Claim 34.

$$\int_0^1 \sin(2k\pi t)\cdot\sin(2\ell\pi(t-\delta))\,\mathrm{d}t = \begin{cases} 0, & \text{if } k\neq\ell \\ \frac{1}{2}\cos(2\ell\pi\delta), & \text{if } k=\ell. \end{cases}$$

Proof of Claim 34.

$$\begin{aligned}\int_0^1 \sin(2k\pi t)\cdot\sin(2\ell\pi(t-\delta))\,\mathrm{d}t &= \int_0^1 \sin(2k\pi t)\cdot\sin(2\ell\pi t)\cos(2\ell\pi\delta)\,\mathrm{d}t \\ &\quad - \int_0^1 \sin(2k\pi t)\cdot\cos(2\ell\pi t)\sin(2\ell\pi\delta)\,\mathrm{d}t \\ &= \cos(2\ell\pi\delta)\int_0^1 \sin(2k\pi t)\cdot\sin(2\ell\pi t)\,\mathrm{d}t,\end{aligned}$$

because, for all $k,\ell\in\{1,2,\dots\}$, Proposition 30 implies

$$\int_0^1 \sin(2k\pi t)\cdot\cos(2\ell\pi t)\,\mathrm{d}t = 0.$$

The proof of our claim follows from Proposition 30 because $\int_0^1 \sin(2k\pi t)\cdot\sin(2\ell\pi t)\,\mathrm{d}t = 1/2$ if (and only if) $k=\ell$; otherwise, it is 0. ◀

Now we can prove Claim 33.

Proof of Claim 33. We simplify the expression for $I_{k,\ell}^{(\delta)}$.

$$\begin{aligned} I_{k,\ell}^{(\delta)} &= \int_0^1 \varphi(kt)\cdot\varphi(\ell(t-\delta))\,\mathrm{d}t \\ &= \frac{16}{\pi^2}\sum_{\text{odd } n>0}\ \sum_{\text{odd } m>0}\frac{1}{mn}\int_0^1 \sin(2n\pi\cdot kt)\sin(2m\pi\cdot\ell(t-\delta))\,\mathrm{d}t \qquad \text{(By Equation 12)}\end{aligned}$$

In light of Claim 34 above, the integral in the RHS survives if and only if $nk=m\ell$. Since $\gcd(k,\ell)=1$, note that $nk=m\ell$ if and only if

$$(n,m)\in J := \Big\{(\ell,k),(3\ell,3k),(5\ell,5k),\dots\Big\}.$$

With this observation and Proposition 30, we get

$$\begin{aligned} I_{k,\ell}^{(\delta)} &= \frac{16}{\pi^2}\sum_{(n,m)\in J}\frac{\cos(2\ell\pi\delta)}{mn}\int_0^1 \sin(2n\pi\cdot kt)\sin(2m\pi\cdot\ell t)\,\mathrm{d}t \\ &= \frac{16}{\pi^2}\sum_{\text{odd } a>0}\frac{\cos(2\ell\pi\delta)}{k\ell\cdot a^2}\int_0^1 \sin(2k\ell a\pi\cdot t)\sin(2k\ell a\pi\cdot t)\,\mathrm{d}t \\ &= \frac{16}{\pi^2}\cdot\frac{1}{k\ell}\sum_{\text{odd } a>0}\frac{1}{a^2}\cdot\frac{\cos(2\ell\pi\delta)}{2} \qquad \text{(By Proposition 30 and Claim 34)} \\ &= \frac{\cos(2\ell\pi\delta)}{k\ell}\cdot\frac{\cancel{8}}{\cancel{\pi^2}}\cdot\frac{\cancel{\pi^2}}{\cancel{8}} = \frac{\cos(2\ell\pi\delta)}{k\ell} \qquad \Big(\text{Because } \sum_{\text{odd } a>0}\frac{1}{a^2} = \frac{3}{4}\cdot\zeta(2) = \frac{\pi^2}{8}\Big)\end{aligned}$$

◀

We are finally ready to prove Lemma 10.

Proof of Lemma 10. Combining Claim 32 and Claim 33, we showed that for relatively prime $k, \ell \in F_p$,

$$I_{k,\ell}^{(\delta)} = \begin{cases} 0 & \text{if } k \cdot \ell \text{ is even} \\ \dfrac{\cos(2\ell\pi \cdot \delta)}{k\ell} & \text{if } k \cdot \ell \text{ is odd} \end{cases}$$

Claim 31 generalizes the result to all $k, \ell \in F_p$ by considering $g = \gcd(k, \ell)$. This proves our lemma that

$$I_{k,\ell}^{(\delta)} = \begin{cases} 0 & \text{if } k \cdot \ell \text{ is even} \\ \dfrac{g^2}{k\ell} \cdot \cos(2\ell\pi \cdot \delta) & \text{if } k \cdot \ell \text{ is odd} \end{cases}$$

◀

C.4 Proof of Corollary 12

Proof of the first part of Corollary 12. The algorithm in Figure 1 declares $\vec{\alpha}$ to be secure either in Step 4 or Step 5.

Suppose our algorithm in Figure 1 declared that Shamir's secret-sharing scheme is secure in Step 4. In this case, $|u| \cdot |v|/g^2$ is even, where $g = \gcd(|u|, |v|)$. Using Corollary 11, we get that our estimation $\varepsilon_{\text{LSB}}^{(\text{est})} = 0$. The relation between our estimation and insecurity yields

$$\varepsilon_{\text{LSB}}(\vec{\alpha}) \leqslant \frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

Suppose our algorithm in Figure 1 declared that Shamir's secret-sharing scheme is secure in Step 5. In this case, $|u| \cdot |v|/g^2 \geqslant \sqrt{p}$ and it is odd. Using Corollary 11, we get that our estimation $\varepsilon_{\text{LSB}}^{(\text{est})} \leqslant 1/\sqrt{p}$. The relation between our estimate and insecurity yields

$$\varepsilon_{\text{LSB}}(\vec{\alpha}) \leqslant \frac{1}{\sqrt{p}} + \frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

◀

Proof of the second part of Corollary 12. We prove that our algorithm outputs "may be insecure" only for an exponentially small fraction of the equivalence classes $[\alpha_1 \colon \alpha_2]$, for distinct evaluation places $\alpha_1, \alpha_2 \in F_p^*$.

First, observe that there are $(p-2)$ equivalence classes $[1\colon 2]$, $[1\colon 3]$, $\dots$, $[1\colon (p-1)]$ (because $\alpha_1 \neq \alpha_2$ and $0 \notin \{\alpha_1, \alpha_2\}$).

Next, let us account for the instances when Figure 1 determines evaluation places $\vec{\alpha}$ may be insecure. Suppose $a = (u/g)$ and $b = (v/g)$, where $g = \gcd(u, v) \in \{1, 2, \dots\}$. We need to upper bound the cardinality of the following set

$$S := \Big\{(a, b) \colon \text{odd } a,\ \text{odd } b,\ \text{and } |a \cdot b| \leqslant \sqrt{p}\Big\}.$$

In this set, for any particular a, the corresponding positive b lies in the set $\{1, 3, 5, \dots, 2n_a - 1\}$, such that $(2n_a - 1)$ is the largest odd number satisfying $a \cdot (2n_a - 1) \leqslant \sqrt{p}$. So, the number of potential odd positive b's is $n_a \leqslant (\sqrt{p} + a)/2a$. As a result, the total number of potential positive and negative candidates is at most $(\sqrt{p}+a)/a$. Let $(2s-1)$ be the largest odd number $\leqslant \sqrt{p}$. Therefore, we have

$$\begin{aligned}\mathsf{card}(S) &\leqslant 2\cdot \sum_{a\in\{1,3,\dots,2s-1\}} \frac{\sqrt{p}+a}{a} = 2\sqrt{p}\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2s-1}\right)+2s\\ &\leqslant 2\sqrt{p}\cdot\left(1+\int_1^s \frac{1}{2t-1}\,\mathrm{d}t\right)+(\sqrt{p}+1)\\ &= \sqrt{p}\cdot\ln(2s-1)+3\sqrt{p}+1 \leqslant \frac{1}{2}\sqrt{p}\cdot\ln p+3\sqrt{p}+1.\end{aligned}$$

Note that for every (a,b), we also counted $(-a,-b)$ in this set; both belong to the same equivalence class. So, every equivalence class is represented at least twice. Therefore, the number of equivalence classes for which our algorithm outputs "may be insecure" is $\leqslant \mathsf{card}(S)/2$. The fraction of equivalence classes that our algorithm declares "may be insecure" is

$$\leqslant \frac{\mathsf{card}(S)/2}{p-2} \leqslant \frac{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p+\frac{3}{2}\cdot\sqrt{p}+\frac{1}{2}}{p-2}.$$

Asymptotically, the upper bound is $\lesssim \frac{1}{4}\cdot\frac{\ln p}{\sqrt{p}}$. Concretely, Appendix C.4.1 proves the upper bound

$$\leqslant \frac{\ln p}{4\sqrt{p}}+\frac{5/2}{\sqrt{p}}$$

for all $p \geqslant 11$. ◀

C.4.1 Proof of inequality used in the proof of Corollary 12

Our objective is to prove the following inequality for primes $p \geqslant 11$.

$$\frac{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p+\frac{3}{2}\cdot\sqrt{p}+\frac{1}{2}}{p-2} \leqslant \frac{\ln p}{4\sqrt{p}}+\frac{5/2}{\sqrt{p}}.$$

We simplify this inequality into a simpler equivalent inequality.

$$\begin{aligned}&\frac{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p+\frac{3}{2}\cdot\sqrt{p}+\frac{1}{2}}{p-2} \leqslant \frac{\ln p}{4\sqrt{p}}+\frac{5/2}{\sqrt{p}}\\ \iff &\cancel{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p}+\cancel{\frac{3}{2}\cdot\sqrt{p}}+\frac{1}{2} \leqslant \cancel{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p}+\frac{\overset{1}{\cancel{5}}}{\cancel{2}}\cdot\sqrt{p}-\frac{1}{2}\cdot\frac{\ln p}{\sqrt{p}}-\frac{5}{\sqrt{p}}\\ \iff &\frac{1}{2}\sqrt{p}+\frac{1}{2}\ln p \leqslant \sqrt{p} \leqslant p-5.\end{aligned}$$

Thus, it suffices to prove the final inequality. Toward this objective, observe that

1. $\ln p \leqslant \sqrt{p}$, for $p \geqslant 2$, and
2. $\sqrt{p} \leqslant p-5$, for $p \geqslant 11$.

Then, for $p \geqslant 11$,

$$\frac{1}{2}\sqrt{p}+\frac{1}{2}\ln p \leqslant \sqrt{p} \leqslant p-5.$$

Therefore,

$$\frac{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p+\frac{3}{2}\cdot\sqrt{p}+\frac{1}{2}}{p-2} \leqslant \frac{\ln p}{4\sqrt{p}}+\frac{5/2}{\sqrt{p}}$$

for all $p \geqslant 11$.

C.5 Proof of Corollary 13

Proof of Corollary 13. Our efficient adversary outputs the s indicated in Theorem 7. After observing the leakage (ℓ_1, ℓ_2), this algorithm performs maximum likelihood decoding – computes whether secret 0 or secret s is more likely to have generated the observed leakage. Then, it predicts the most likely of the two events.

We emphasize that the secret $s' \in F^*$ that witnesses the maximum statistical distance between the leakage distributions $\mathrm{LSB}(\mathsf{Share}(0))$ and $\mathrm{LSB}(\mathsf{Share}(s'))$ may be different from the secret s defined above. Secret $s \in F^*$ witnesses the maximum *estimate* of the statistical distance between the distributions $\mathrm{LSB}(\mathsf{Share}(0))$ and $\mathrm{LSB}(\mathsf{Share}(s))$.

For brevity, define $\mathrm{err} := \frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}$. Given $\vec{\alpha}$, we run the LLL algorithm [39] to obtain $(u, v) \in [\alpha_1 \colon \alpha_2]$ such that $|u|_p, |v|_p \leqslant B$, where $B = \lceil 8^{1/4} \cdot \sqrt{p} \rceil$. Define $g = \gcd(|u|_p, |v|_p)$.

We are given that $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) > 2 \cdot \mathrm{err}$. We claim that $\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha}) > \mathrm{err}$ and $|u|_p \cdot |v|_p / g^2$ is odd. Suppose not; then, there are two possibilities.

1. If $|u|_p \cdot |v|_p / g^2$ is even. In this case, $\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha}) = 0$ and, hence, $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \leqslant \mathrm{err}$, by Corollary 11; a contradiction.
2. If $\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha}) \leqslant \mathrm{err}$ and $|u|_p \cdot |v|_p / g^2$ is odd. In this case, $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) \leqslant 2 \cdot \mathrm{err}$, by Corollary 11; a contradiction.

So, the signs of $\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha})$ and $\left(\frac{1}{p}\Sigma^{(0)}_{\alpha_1,\alpha_2} - \frac{1}{p} \cdot \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}\right)$ are identical (by Claim 35). Using this property, Appendix C.6 proves that the advantage of the maximum likelihood decoder is

$$\geqslant \varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha}) - \mathrm{err} \geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) - 2 \cdot \mathrm{err}.$$

◀

C.6 Alternative Proof for Corollary 13

▷ Claim 35. For $\mathsf{ShamirSS}(2, 2, \vec{\alpha} = (\alpha_1, \alpha_2))$ and secret $s \in F$, define $\mathrm{err} := \frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}$. Consider $|\alpha_1|_p, |\alpha_2|_p < \lceil 8^{1/4}\sqrt{p} \rceil$ and $|\alpha_1|_p \cdot |\alpha_2|_p / g^2$ is odd with $g = \gcd(|\alpha_1|_p, |\alpha_2|_p)$. When $\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) > 2 \cdot \mathrm{err}$,

$$\mathrm{sign}\left(\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha})\right) = \mathrm{sign}\left(\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p}\right)$$

where $\Delta := (s \cdot 2^{-1}) \cdot (\alpha_1^{-1} - \alpha_2^{-1}) \in F$.

Proof of Claim 35. By Lemma 9 and $\delta = \frac{\mathrm{sign}_p(\Delta)|\Delta|_p}{p}$,

$$\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p} = \mathrm{sign}_p(\alpha_1) \cdot \mathrm{sign}_p(\alpha_2) \cdot \left(I^{(0)}_{|\alpha_1|_p,|\alpha_2|_p} - I^{(\delta)}_{|\alpha_1|_p,|\alpha_2|_p}\right) \pm 2 \cdot \frac{4(|\alpha_1|_p + |\alpha_2|_p) - (3/2)}{p}$$

which we can rewrite equivalently as

$$\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2} - \Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p} \pm 2 \cdot \frac{4(|\alpha_1|_p + |\alpha_2|_p) - (3/2)}{p} = \mathrm{sign}_p(\alpha_1) \cdot \mathrm{sign}_p(\alpha_2) \cdot \left(I^{(0)}_{|\alpha_1|_p,|\alpha_2|_p} - I^{(\delta)}_{|\alpha_1|_p,|\alpha_2|_p}\right).$$

For $|\alpha_1|_p, |\alpha_2|_p < \lceil 8^{1/4}\sqrt{p}\rceil$,

$$2\cdot\frac{4(|\alpha_1|_p+|\alpha_2|_p)-(3/2)}{p} \leqslant 2\cdot \mathrm{err} < \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) = \frac{\left|\Sigma^{(0)}_{\alpha_1,\alpha_2}-\Sigma^{(\Delta)}_{\alpha_1,\alpha_2}\right|}{p}.$$

which implies that $\pm 2\cdot\frac{4(|\alpha_1|_p+|\alpha_2|_p)-(3/2)}{p}$ does not change the sign of $\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2}-\Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p}$. Hence,

$$\mathrm{sign}\left(\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2}-\Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p}\pm 2\cdot\frac{4(|\alpha_1|_p+|\alpha_2|_p)-(3/2)}{p}\right) = \mathrm{sign}\left(\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2}-\Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p}\right)$$

and therefore,

$$\begin{aligned}
&\mathrm{sign}\left(\frac{\Sigma^{(0)}_{\alpha_1,\alpha_2}-\Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p}\right)\\
&\quad= \mathrm{sign}\Big(\mathrm{sign}_p(\alpha_1)\cdot\mathrm{sign}_p(\alpha_2)\cdot\Big(I^{(0)}_{|\alpha_1|_p,|\alpha_2|_p}-I^{(\delta)}_{|\alpha_1|_p,|\alpha_2|_p}\Big)\Big)\\
&\quad= \mathrm{sign}\left(\mathrm{sign}_p(\alpha_1)\cdot\mathrm{sign}_p(\alpha_2)\cdot\left(\sin^2(|v|_p\pi\cdot\delta)\cdot\frac{g^2}{|u|_p\cdot|v|_p}\right)\right) && \text{(By Lemma 10)}\\
&\quad= \mathrm{sign}\big(\mathrm{sign}_p(\alpha_1)\cdot\mathrm{sign}_p(\alpha_2)\big) && \left(\text{Since } \sin^2(|v|_p\pi\cdot\delta)\cdot\frac{g^2}{|u|_p\cdot|v|_p}>0\right)\\
&\quad= \mathrm{sign}\Big(\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha})\Big)
\end{aligned}$$

◀

Alternative Proof for Corollary 13. For any secret $s\in F$, let us first define the distinguishing advantage of the maximum likelihood decoder as

$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha};s) := \frac{\Sigma^{(0)}_{\alpha_1,\alpha_2}-\Sigma^{(\Delta)}_{\alpha_1,\alpha_2}}{p}$$

where $\Delta := (s\cdot 2^{-1})\cdot(\alpha_1^{-1}-\alpha_2^{-1})\in F$ and the estimate $\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha};s)\in[0,1]$ satisfying

$$\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha};s) = \varepsilon_{\mathrm{LSB}}(\vec{\alpha};s) \qquad \pm\,\mathrm{err}$$

where $\mathrm{err} := \frac{8^{5/4}}{\sqrt{p}}+\frac{13/2}{p}$. Given Claim 35, we know that for any secret $s\in F$,

$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha};s) \geqslant \varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha};s)-\mathrm{err}. \tag{14}$$

and

$$\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha};s) \geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha};s)-\mathrm{err}. \tag{15}$$

Consider secret $s^*\in F$ that achieves the maximum $\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha};s)$. Define $\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha};s^*)$ as

$$\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha}) := \max_{s\in F}\varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha};s) = \varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha};s^*).$$

Similarly, consider $\tilde{s}^*\in F$ that reaches maximum $\varepsilon_{\mathrm{LSB}}(\vec{\alpha};s)$, and define $\varepsilon_{\mathrm{LSB}}(\vec{\alpha};s^*)$ as

$$\varepsilon_{\mathrm{LSB}}(\vec{\alpha}) := \max_{s\in F}\varepsilon_{\mathrm{LSB}}(\vec{\alpha};s) = \varepsilon_{\mathrm{LSB}}(\vec{\alpha};\tilde{s}^*).$$

Then,

$$\begin{aligned}
\varepsilon_{\mathrm{LSB}}(\vec{\alpha}; s^*) &\geqslant \varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha}; s^*) - \mathrm{err} = \varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha}) - \mathrm{err} && \text{(By Equation 14)}\\
&\geqslant \varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha}; \tilde{s}^*) - \mathrm{err} && \text{(Because } \varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha}; s^*) = \max_{s \in F} \varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha}; s))\\
&\geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha}; \tilde{s}^*) - 2 \cdot \mathrm{err} && \text{(By Equation 15)}\\
&= \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) - 2 \cdot \mathrm{err} > 0
\end{aligned}$$

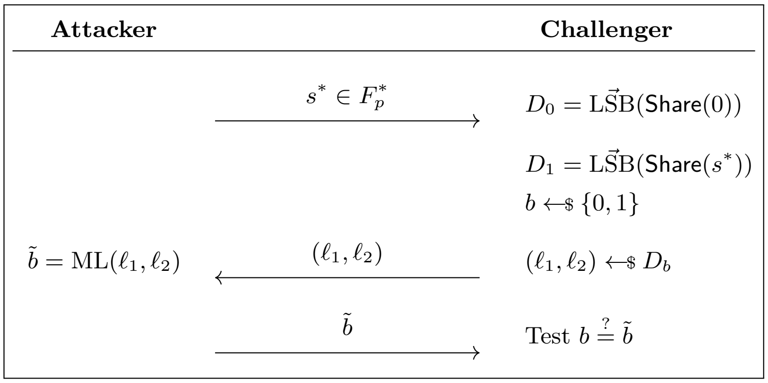
and therefore, the distinguishing advantage of the maximum likelihood decoder is

$$\geqslant \varepsilon^{(\mathrm{est})}_{\mathrm{LSB}}(\vec{\alpha}) - \mathrm{err} \geqslant \varepsilon_{\mathrm{LSB}}(\vec{\alpha}) - 2 \cdot \mathrm{err}.$$

◀

C.7 Efficient Distinguisher Construction

Consider the following security game (illustrated in the figure below). The attacker picks a secret $s \in F_p^*$ and sends it to the challenger. The challenger picks a random bit $b \in \{0, 1\}$. If $b = 0$, the challenger samples (ℓ_1, ℓ_2) from distribution $D_0 := \mathrm{L\vec{S}B}(\mathsf{Share}(0))$ and sends it to the attacker. Otherwise, the challenger samples (ℓ_1, ℓ_2) from distribution $D_1 := \mathrm{L\vec{S}B}(\mathsf{Share}(s))$ and sends it to the attacker. The attacker aims to guess which distribution (ℓ_1, ℓ_2) is sampled from. It uses the maximum likelihood decoder and then returns its guess $\tilde{b}$ to the challenger. The attacker wins the security game if $b = \tilde{b}$.


Attacker
Challenger
$s^* \in F_p^*$
$D_0 = \mathrm{L\vec{S}B}(\mathsf{Share}(0))$
$D_1 = \mathrm{L\vec{S}B}(\mathsf{Share}(s^*))$
$b \leftarrow\$ \{0, 1\}$
$\tilde{b} = \mathrm{ML}(\ell_1, \ell_2)$
(ℓ_1, ℓ_2)
$(\ell_1, \ell_2) \leftarrow\$ D_b$
$\tilde{b}$
Test $b \overset{?}{=} \tilde{b}$

The maximum likelihood distinguisher outputs

$$\tilde{b} = \begin{cases} 0 & \text{if } \Pr[(\ell_1, \ell_2)|s = 0] \geqslant \Pr[(\ell_1, \ell_2)|s = s^*] \\ 1 & \text{if } \Pr[(\ell_1, \ell_2)|s = 0] < \Pr[(\ell_1, \ell_2)|s = s^*] \end{cases}$$

In other words, the output depends on $\mathrm{sign}(\Pr[(\ell_1, \ell_2)|s = 0] - \Pr[(\ell_1, \ell_2)|s = s^*])$.

For evaluation places (u, v), where $|u| \cdot |v|/g^2$ is odd and $g = \gcd(|u|, |v|)$, and $\Delta = (s^* \cdot 2^{-1}) \cdot (u^{-1} - v^{-1}) \in F^*$, we get

$$\begin{aligned}
&\Pr[(\ell_1, \ell_2)|s = 0] - \Pr[(\ell_1, \ell_2)|s = s^*]\\
&= (-1)^{\ell_1+\ell_2} \cdot \frac{\Sigma^{(0)}_{u,v} - \Sigma^{(\Delta)}_{u,v}}{4p} && \text{(Appendix C.1)}\\
&= \frac{(-1)^{\ell_1+\ell_2} \cdot \mathrm{sign}_p(u) \cdot \mathrm{sign}_p(v)}{4} \cdot \left(I^{(0)}_{|u|,|v|} - I^{(\delta)}_{|u|,|v|} \pm 2 \cdot \frac{4(|u| + |v|) - (3/2)}{p} \right)\\
& && \text{(Lemma 9 and } \delta = \frac{\mathrm{sign}_p(\Delta)|\Delta|_p}{p})
\end{aligned}$$

$$= \frac{(-1)^{\ell_1+\ell_2}\cdot \mathrm{sign}_p(u)\cdot \mathrm{sign}_p(v)}{4}\cdot\left(\sin^2(|v|\pi\cdot\delta)\cdot\frac{g^2}{|u|\cdot|v|}\pm 2\cdot\frac{4(|u|+|v|)-(3/2)}{p}\right)$$

(Lemma 10)

Consider attacker picks $s = \pm(u^{-1}\cdot v - 1)^{-1} \in F^*$ such that

$$\begin{aligned}&\Pr[(\ell_1,\ell_2)|s=0]-\Pr[(\ell_1,\ell_2)|s=s^*]\\ &= \frac{(-1)^{\ell_1+\ell_2}\cdot \mathrm{sign}_p(u)\cdot \mathrm{sign}_p(v)}{4}\cdot\left(\cos^2(\pi/2p)\cdot\frac{g^2}{|u|\cdot|v|}\pm 2\cdot\frac{4(|u|+|v|)-(3/2)}{p}\right)\end{aligned}$$

Since $\mathsf{SD}\big(\mathrm{LSB}(\mathsf{Share}(0))\ ,\ \mathrm{LSB}(\mathsf{Share}(s))\big) > \frac{4(|u|+|v|)-(3/2)}{p}$ by our assumption, then

$$\cos^2(\pi/2p)\cdot\frac{g^2}{|u|\cdot|v|} - 2\cdot\frac{4(|u|+|v|)-(3/2)}{p} > 0.$$

Hence,

$$\mathrm{sign}(\Pr[(\ell_1,\ell_2)|s=0]-\Pr[(\ell_1,\ell_2)|s=s^*]) = (-1)^{\ell_1+\ell_2}\cdot \mathrm{sign}_p(u)\cdot \mathrm{sign}_p(v).$$

There exists an efficient maximum likelihood distinguisher computing $(-1)^{\ell_1+\ell_2}\cdot \mathrm{sign}_p(u)\cdot \mathrm{sign}_p(v)$. If $(-1)^{\ell_1+\ell_2}\cdot \mathrm{sign}_p(u)\cdot \mathrm{sign}_p(v) > 0$, then the maximum likelihood distinguisher outputs $\tilde{b}=0$. Otherwise, it outputs $\tilde{b}=1$.

D Equivalence classes for Evaluation Places

Consider Shamir's secret-sharing scheme among n parties with reconstruction threshold k over the prime field F_p of order $p \geqslant 3$. The secret-sharing scheme is the Massey secret-sharing scheme [48] corresponding to the (punctured) Reed-Solomon code with evaluation places $(0,\alpha_1,\alpha_2,\dots,\alpha_n)$. That is, the dealer chooses a random F_p-polynomial $P(Z)$ of degree $< k$ conditioned on $P(Z=0)$ being the secret s. Evaluating this polynomial at evaluation places $Z=\alpha_1,\alpha_2,\dots,\alpha_n$ generates the secret shares $s_1,s_2,\dots,s_n$, respectively.

▶ **Lemma 36** (Equivalence Classes of Evaluation Places)**.** *The (punctured) Reed-Solomon code corresponding to evaluation places $(0,\alpha_1,\alpha_2,\dots,\alpha_n)$ is identical to the (punctured) Reed-Solomon code corresponding to evaluation places $(0,\Lambda\cdot\alpha_1,\Lambda\cdot\alpha_2,\dots,\Lambda\cdot\alpha_n)$, for any $\Lambda \in F_p^*$.*

This proposition is a consequence of the properties of Generalized Reed-Solomon codes [28, 40] (see Appendix F.2.1 for definition and one of its primitive properties). In particular, since the linear codes are identical, the corresponding Massey secret-sharing schemes have identical resilience/vulnerability to attacks. That is, two secret-sharing schemes

$$\mathsf{ShamirSS}(n,k,(\alpha_1,\alpha_2,\dots,\alpha_n)) \quad\text{and}\quad \mathsf{ShamirSS}(n,k,(\Lambda\cdot\alpha_1,\Lambda\cdot\alpha_2,\dots,\Lambda\cdot\alpha_n))$$

have identical resilience/vulnerability to attacks, for any $\Lambda \in F_p^*$. Therefore, for given distinct evaluation places $\alpha_1,\alpha_2,\dots,\alpha_n \in F_p^*$, we define the equivalence class

$$[\alpha_1\colon\alpha_2\colon\cdots\colon\alpha_n] := \big\{(\Lambda\cdot\alpha_1,\Lambda\cdot\alpha_2,\dots,\Lambda\cdot\alpha_n)\colon \Lambda\in F_p^*\big\}.$$

Determining the security of the evaluation places $(\alpha_1,\dots,\alpha_n)$ is equivalent to determining the security of *any element* in the equivalence class $[\alpha_1\colon\cdots\colon\alpha_n]$.

E Security against Physical Bit Leakage Corollaries

We consider $\mathsf{ShamirSS}(n = 2, k = 2, (\alpha_1, \alpha_2))$ over the prime field F of order $p \geqslant 3$. This section considers p a Mersenne or Fermat prime, i.e., $p = 2^\lambda \pm 1$, where λ is the security parameter.

E.1 Proof of Proposition 14

The following proposition will be used without proof.

▶ **Proposition 37.** *Let F be a prime field of order $p = 2^\lambda - 1$. Suppose $x \in F$ and define $x' = x \cdot (2^i) \in F$, where $i \in \{-\lambda + 1, \ldots, 0, 1, \ldots, \lambda - 1\}$. Then, the binary representation of x' is a cyclic left rotation of the binary representation of x by i bits.*

We clarify that if i is negative, then "i bit cyclic left rotation" is the same as "$|i|$ bit cyclic right rotation." This proposition is straightforward from the identity that $2^\lambda = 1 \mod p$.

Proof of Proposition 14. Let λ be the security parameter.

- **Case 1:** $p = 2^\lambda - 1$ is a Mersenne prime.
  Then for all $i \in \{0, 1, 2, \ldots, \lambda - 1\}$, we have $p = -1 \mod 2^{i+1}$.
  Note that, by Proposition 37, if $\mathrm{PHYS}_i(x) = 0$, then $\mathrm{PHYS}_0(x \cdot 2^{-i}) = 0$; similarly, if $\mathrm{PHYS}_i(x) = 1$, then $\mathrm{PHYS}_0(x \cdot 2^{-i}) = 1$. Therefore, $\mathrm{PHYS}_i(x) = \mathrm{PHYS}_0(x \cdot 2^{-i})$.
- **Case 2:** $p = 2^\lambda + 1$ is a Fermat prime.
  Then for all $i \in \{0, 1, 2, \ldots, \lambda - 1\}$, $p = 1 \mod 2^{i+1}$. Let F be a prime field of order $p = 2^\lambda + 1$. Then, for $x \in F$, $\mathrm{PHYS}_i(2x + 1) = \mathrm{PHYS}_{i-1}(x)$. Therefore,

  $$\mathrm{PHYS}_i(x) = \mathrm{PHYS}_{i-1}\left(\frac{x-1}{2}\right) = \mathrm{PHYS}_{i-2}\left(\frac{x-3}{4}\right) = \cdots = \mathrm{PHYS}_0\left(\frac{x - 2^i + 1}{2^i}\right).$$

  Upon Simplification, we can conclude $\mathrm{PHYS}_i(x) = \mathrm{PHYS}_0\big(2^{-i}x + 2^{-i} - 1\big)$.

◀

E.2 Reduction: Leakage attack when $2^k\alpha_1 = \alpha_2$

This section provides the detailed calculations behind the reduction from joint distribution of physical-bit leakages to of LSBs, illustrated in Section 8.1, which leads to Lemma 15.

▶ **Proposition 38.** *Let F be the prime field of order $p = 2^\lambda \pm 1$. Let $\alpha_1, \alpha_2 \in F$ such that $\alpha_1 \neq \alpha_2$ yet $2^k\alpha_1 = \alpha_2$ for some $k \in \{0, 1, \ldots, \lambda - 1\}$. Given $s \in F$, for uniformly random $u \in F$, there exists $t^* \in F$ that makes the two following joint distributions equivalent.*

$$(\mathrm{PHYS}_i(s + u\alpha_1), \mathrm{PHYS}_j(s + u\alpha_2)) \equiv (\mathrm{PHYS}_0(1 + t^* + v\alpha_1), \mathrm{PHYS}_0(t^* + v\alpha_2))$$

where $v := v(u)$ is a uniform distribution i.i.d. to the distribution of u.

Proof of Proposition 38. We divide it into two cases.

- **Case 1:** $p = 2^\lambda - 1$.
  Applying Proposition 14 and substituting $t := s2^{-j}$, $v := u2^{-j}$, and $k := i - j$ gives us

  $$\begin{aligned}&(\mathrm{PHYS}_i(s + u\alpha_1), \mathrm{PHYS}_j(s + u\alpha_2))\\ &\equiv \big(\mathrm{PHYS}_0((s + u\alpha_1) \cdot 2^{-i}), \mathrm{PHYS}_0((s + u\alpha_2) \cdot 2^{-j})\big) && \text{(By Proposition 14)}\end{aligned}$$

$$\equiv \left(\mathrm{PHYS}_0(s \cdot 2^{-i} + u\alpha_1 \cdot 2^{-i}), \mathrm{PHYS}_0(s \cdot 2^{-j} + u\alpha_2 \cdot 2^{-j})\right)$$
$$\equiv \left(\mathrm{PHYS}_0(t \cdot 2^{k} + \alpha_1 \cdot v2^{k}), \mathrm{PHYS}_0(t + \alpha_2 \cdot v)\right) \qquad \text{(By aforesaid substitutions)}$$

Let $t^* := (2^k - 1)^{-1}$. Then $t^*(2^k - 1) = 1$ which simplifies to $t^* 2^k = t^* + 1$. Hence, by substituting t^* for t in above equation, we conclude

$$(\mathrm{PHYS}_i(s + u\alpha_1), \mathrm{PHYS}_j(s + u\alpha_2)) \equiv (\mathrm{PHYS}_0(1 + t^* + v\alpha_1), \mathrm{PHYS}_0(t^* + v\alpha_2))$$

- **Case 2:** $p = 2^\lambda + 1$.
  We again apply Proposition 14, but for this case we substitute $t := (s+1)2^{-j}$, $v := u2^{-j}$ and $k := i - j$, and choose $t^* := (2^k - 1)^{-1} - 1$.

$$(\mathrm{PHYS}_i(s + u\alpha_1), \mathrm{PHYS}_j(s + u\alpha_2))$$
$$\equiv \left(\mathrm{PHYS}_0(s2^{-i} + u\alpha_1 2^{-i} + (2^{-i} - 1)), \mathrm{PHYS}_0(s2^{-j} + u\alpha_2 2^{-j} + (2^{-j} - 1))\right)$$
$$\equiv \left(\mathrm{PHYS}_0(2^{-i}(s+1) + u\alpha_1 2^{-i} - 1), \mathrm{PHYS}_0(2^{-j}(s+1) + u\alpha_2 2^{-j} - 1)\right)$$
$$\equiv \left(\mathrm{PHYS}_0(2^k t + \alpha_1 \cdot v2^k - 1), \mathrm{PHYS}_0(t + \alpha_2 v - 1)\right) \qquad \text{(By aforesaid substitutions)}$$
$$\equiv (\mathrm{PHYS}_0(1 + t^* + v\alpha_1), \mathrm{PHYS}_0(t^* + v\alpha_2)) \qquad \text{(Substitute } t^* \text{ for } t\text{)}$$

As F is a prime field, the mapping $u \mapsto v := u2^{-j}$ an automorphism over F, making the distribution of v also uniform. ◀

Lemma 15 is then a direct consequence of Proposition 38, so Lemma 15 is also proved.

E.3 Proof of Corollary 16

We prove Corollary 16 separately for p is a Mersenne or Fermat prime.

E.3.1 Case A: p is a Mersenne Prime

Due to the properties of the F, when $p = 2^\lambda - 1$ is a Mersenne prime, we can reduce arbitrary physical bit attacks on $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ to LSB leakage attacks on $\mathsf{ShamirSS}(2, 2, \vec{\alpha}')$, for an appropriately defined $\vec{\alpha}'$.

▶ **Lemma 39.** *Let F_p be a prime field of order $p = 2^\lambda - 1$. Consider evaluation places $\alpha_1, \alpha_2 \in F_p^*$ such that $2^k \cdot \alpha_1 \neq \alpha_2$, for all $k \in \{0, 1, \ldots, \lambda - 1\}$. Consider the leakage attack $\vec{\mathrm{PHYS}}_{i,j}$ for any $i, j \in \{0, 1, \ldots, \lambda - 1\}$. Define $\alpha_1' = 2^{-i} \cdot \alpha_1$ and $\alpha_2' = 2^{-j} \cdot \alpha_2$. For any $s \in F_p$, let D denote the joint leakage distribution generated by the leakage function $\vec{\mathrm{PHYS}}_{i,j}$ when the secret shares are generated using the $\mathsf{ShamirSS}(2, 2, \vec{\alpha})$ secret-sharing scheme. Likewise, D' denotes the joint leakage distribution generated by the leakage function $\vec{\mathrm{LSB}}$ when the secret shares are generated using the $\mathsf{ShamirSS}(2, 2, \vec{\alpha}')$ secret-sharing scheme instead. Then, the distributions D and D' are identical.*

Since $2^k \cdot \alpha_1 \neq \alpha_2$, for all $k \in \{0, 1, \ldots, \lambda - 1\}$, we conclude that $\alpha_1' \neq \alpha_2'$, for all $i, j \in \{0, 1, \ldots, \lambda - 1\}$. Therefore, the secret-sharing scheme $\mathsf{ShamirSS}(2, 2, \vec{\alpha}')$ is valid. We prove that the distributions D and D' are identical, for all $s \in F_p$, using Proposition 14.

Proof of Lemma 39. Given $s \in F_p$,

$$2\mathsf{SD}\left(\vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(s))\right)$$
$$= \sum_{\vec{\ell} \in \{0,1\}^2} \left| \Pr\left[\vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(0)) = \vec{\ell}\right] - \Pr\left[\vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(s)) = \vec{\ell}\right] \right|$$

$$\begin{aligned}
&= \sum_{\vec{\ell}\in\{0,1\}^2} \Bigg| \mathop{\mathbb{E}}_{x}\Big[\mathbb{1}_{\mathrm{PHYS}_i^{-1}(\ell_1)}(\alpha_1 x)\cdot \mathbb{1}_{\mathrm{PHYS}_j^{-1}(\ell_2)}(\alpha_2 x)\Big] \\
&\qquad\qquad - \mathop{\mathbb{E}}_{x}\Big[\mathbb{1}_{\mathrm{PHYS}_i^{-1}(\ell_1)}(\alpha_1 x + s)\cdot \mathbb{1}_{\mathrm{PHYS}_j^{-1}(\ell_2)}(\alpha_2 x + s)\Big]\Bigg| \\
&= \sum_{\vec{\ell}\in\{0,1\}^2} \Bigg| \mathop{\mathbb{E}}_{x}\Big[\mathbb{1}_{\mathrm{PHYS}_i^{-1}(0)}(\alpha_1 x)\cdot \mathbb{1}_{\mathrm{PHYS}_j^{-1}(0)}(\alpha_2 x)\Big] \\
&\qquad\qquad - \mathop{\mathbb{E}}_{x}\Big[\mathbb{1}_{\mathrm{PHYS}_i^{-1}(0)}(\alpha_1 x + s)\cdot \mathbb{1}_{\mathrm{PHYS}_j^{-1}(0)}(\alpha_2 x + s)\Big]\Bigg| \\
&\qquad\qquad\qquad\qquad\qquad \left(\text{Because } \mathbb{1}_{\mathrm{PHYS}_k^{-1}(1)} = 1 - \mathbb{1}_{\mathrm{PHYS}_k^{-1}(0)}\right) \\
&= 4\cdot \Bigg| \mathop{\mathbb{E}}_{x}\big[\mathbb{1}_E(2^{-i}\alpha_1 x)\cdot \mathbb{1}_E(2^{-j}\alpha_2 x)\big] \\
&\qquad\qquad - \mathop{\mathbb{E}}_{x}\big[\mathbb{1}_E(2^{-i}\alpha_1 x + 2^{-i}s)\cdot \mathbb{1}_E(2^{-j}\alpha_2 x + 2^{-j}s)\big]\Bigg| \\
&\qquad\qquad\qquad\qquad\qquad \left(\text{By Proposition 14, } E := \mathrm{PHYS}_0^{-1}(0)\right) \\
&= 4\cdot \Bigg| \mathop{\mathbb{E}}_{x}\big[\mathbb{1}_E(2^{-i}\alpha_1 x)\cdot \mathbb{1}_E(2^{-j}\alpha_2 x)\big] \\
&\qquad\qquad - \mathop{\mathbb{E}}_{y}\big[\mathbb{1}_E(2^{-i}\alpha_1 \cdot y)\cdot \mathbb{1}_E(2^{-j}\alpha_2 \cdot y + s')\big]\Bigg| \qquad (16)
\end{aligned}$$

where $y := \left(x + \alpha_1^{-1}\cdot s\right)$ and $s' := \left((1-\alpha_1^{-1}\alpha_2)\cdot 2^{-j}s\right)$. These substitutions are automorphisms over F^*.

Observe that Equation 16 equals $\varepsilon_{\mathrm{LSB}}(2^{-i}\alpha_1, 2^{-j}\alpha_2)$. Therefore, we conclude that the insecurity of $\mathsf{ShamirSS}(2,2,(\alpha_1,\alpha_2))$ secret-sharing scheme against the $\overrightarrow{\mathrm{PHYS}}_{i,j}$ is identical to the insecurity of the $\mathsf{ShamirSS}(2,2,(2^{-i}\alpha_1, 2^{-j}\alpha_2))$ secret-sharing scheme against the LSB attack. ◀

Before we start proving Corollary 16 for Mersenne primes, we first prove the following two corollaries on the estimated insecurity of $\mathsf{ShamirSS}(2,2,\vec{\alpha})$.

▶ **Corollary 40.** *Let F_p be the prime field of order $p = 2^\lambda - 1$. Consider distinct evaluation places $\vec{\alpha} = (\alpha_1, \alpha_2)$ and the corresponding secret-sharing scheme* $\mathsf{ShamirSS}(2,2,\vec{\alpha})$*. Define*

$$\varepsilon_{\mathrm{PHYS}}^{(\mathrm{est})} = \begin{cases} 1, & \text{if } 2^t\cdot\alpha_1 = \alpha_2 \\ & \text{for some } t\in\{0,1,\dots,\lambda-1\}, \\ \max\limits_{k\in\{0,1,\dots,p-1\}} \varepsilon_{\mathrm{LSB}}^{(\mathrm{est})}\big(\,(2^k\alpha_1,\alpha_2)\,\big), & \text{if } 2^t\cdot\alpha_1 \neq \alpha_2 \\ & \text{for all } t\in\{0,1,\dots,\lambda-1\}. \end{cases}$$

Then,

$$\varepsilon_{\mathrm{PHYS}}^{(\mathrm{est})}(\vec{\alpha}) = \varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \quad \pm\left(\frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}\right).$$

Proof of Corollary 40. If $2^t\cdot\alpha_1 = \alpha_2$, for some $t\in\{0,1,\dots,\lambda-1\}$, we have $\varepsilon_{\mathrm{PHYS}}^{(\mathrm{est})}(\vec{\alpha}) = 1$. Lemma 15 presents a physical bit leakage attack with distinguishing advantage $1 - 1/p$; therefore, $\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \geqslant 1 - 1/p$. So, we conclude that

$$\varepsilon_{\mathrm{PHYS}}^{(\mathrm{est})}(\vec{\alpha}) = \varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \pm \frac{1}{p}.$$

We are left with the case where $2^t\alpha_1 \neq \alpha_2$ for all $t \in \{0,1,\dots,\lambda-1\}$. Lemma 39 shows that the leakage distribution of $\text{PHYS}_{i,j}$ on $\mathsf{ShamirSS}(2,2,\vec{\alpha})$ is identical to the leakage distribution $\vec{\text{LSB}}$ on $\mathsf{ShamirSS}(2,2,\ (2^{-i}\alpha_1, 2^{-j}\alpha_j)\ )$.

Recall that the secret-sharing scheme $\mathsf{ShamirSS}(2,2,\ (2^{-i}\alpha_1, 2^{-j}\alpha_j)\ )$ is identical to the secret-sharing scheme $\mathsf{ShamirSS}(2,2,\ (2^{j-i}\alpha_1, \alpha_j)\ )$, by Lemma 36 in Appendix D. Hence,

$$\varepsilon_{\text{PHYS}}(\vec{\alpha}) = \max_{t\in\{0,1,\dots\}} \varepsilon_{\text{LSB}}\big(\ (2^t\alpha_1,\alpha_2)\ \big).$$

We know that our estimation $\varepsilon^{(\text{est})}_{\text{LSB}}(\cdot)$ is a tight estimation of $\varepsilon_{\text{LSB}}(\cdot)$ due to Corollary 11. Therefore, we conclude that

$$\varepsilon^{(\text{est})}_{\text{PHYS}}(\vec{\alpha}) = \varepsilon_{\text{PHYS}}(\vec{\alpha}) \quad \pm \left(\frac{8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}\right).$$

◀

▶ **Corollary 41.** *Let F_p be the prime field with order $p = 2^\lambda - 1$. Consider distinct evaluation places $\vec{\alpha} = (\alpha_1,\alpha_2)$ and the corresponding $\mathsf{ShamirSS}(2,2,\vec{\alpha})$ over F_p. If $\varepsilon_{\text{PHYS}}(\vec{\alpha}) > \frac{2\cdot 8^{5/4}}{\sqrt{p}} + \frac{13}{p}$, then there is an efficient algorithm that generates $(s,f) \in F_p^* \times \text{PHYS}$ and can distinguish the secret 0 from the secret s with an advantage*

$$\geqslant \varepsilon_{\text{PHYS}}(\vec{\alpha}) \quad - \frac{2\cdot 8^{5/4}}{\sqrt{p}} - \frac{13}{p}$$

by leaking f from the secret shares.

Proof of Corollary 41. If there is $t \in \{0,1,\dots,\lambda-1\}$ such that $2^t\alpha_1 = \alpha_2$, then Lemma 15 presents an explicit leakage attack that suffices for this corollary.

If there $2^t\alpha_1 \neq \alpha_2$ for all $t \in \{0,1,\dots,\lambda-1\}$, then Lemma 39 helps relate physical bit attacks and LSB attacks. Suppose t is the witness such that $\varepsilon^{(\text{est})}_{\text{PHYS}}(\vec{\alpha}) = \varepsilon^{(\text{est})}_{\text{LSB}}(\ (2^t\alpha_1,\alpha_2)\ )$. Then, consider the adversary against $\mathsf{ShamirSS}(2,2,\ (2^t\alpha_1,\alpha_2)\ )$ that uses the LSB attack as guaranteed by Corollary 13. Lemma 39 proves that the leakage distribution of the physical bit attack $\vec{\text{PHYS}}_{0,t}$ on $\mathsf{ShamirSS}(2,2,\vec{\alpha})$ secret-sharing scheme has an identical distribution. So, we run the adversary of Corollary 13 by leaking $\vec{\text{PHYS}}_{0,t}$ from the secret shares of the $\mathsf{ShamirSS}(2,2,\vec{\alpha})$ secret-sharing scheme. ◀

Finally, we prove Corollary 16 for Mersenne primes.

Proof of Corollary 16, for Mersenne primes. Proof of the first part. If the algorithm in Figure 2 determined (α_1,α_2) to be secure, then the algorithm in Figure 1 determined $(2^t\alpha_1,\alpha_2)$ to be secure, for all $t \in \{0,1,\dots,\lambda-1\}$. For $t \in \{0,1,\dots,\lambda-1\}$, by Corollary 12, we get the bound that

$$\varepsilon_{\text{LSB}}\big(\ (2^t\alpha_1,\alpha_2)\ \big) \leqslant \frac{1+8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

Just like the proof of Corollary 40, we have

$$\varepsilon_{\text{PHYS}}(\vec{\alpha}) = \max_{t\in\{0,1,\dots,\lambda-1\}} \varepsilon_{\text{LSB}}\big(\ (2^t\alpha_1,\alpha_2)\ \big) \leqslant \frac{1+8^{5/4}}{\sqrt{p}} + \frac{13/2}{p}.$$

Proof of the second part. If the algorithm in Figure 2 outputs "may be insecure" then there is some $k \in \{0,1,\dots,\lambda-1\}$ such that the algorithm in Figure 1 outputs "may be insecure” for $(2^k\alpha_1, \alpha_2)$. Corollary 12 proves that the algorithm in Figure 1 outputs “may be insecure” for at most

$$\frac{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p + \frac{3}{2}\cdot\sqrt{p} + \frac{1}{2}}{p-2}$$

fraction of the equivalence classes. By, a union bound over $k \in \{0, 1, \ldots, \lambda - 1\}$, Figure 2 outputs “may be insecure” for at most

$$\log_2 p \cdot \frac{\frac{1}{4}\cdot\sqrt{p}\cdot\ln p + \frac{3}{2}\cdot\sqrt{p} + \frac{1}{2}}{p-2}$$

fraction of the equivalence classes. ◀

E.3.2 Case B: p is a Fermat Prime

We start by extending Lemma 8 from $\vec{\mathrm{LSB}}$ to $\vec{\mathrm{PHYS}}_{i,j}$, over Fermat primes.

▶ **Lemma 42.** *Fix prime $p = 2^\lambda + 1$. Consider* $\mathsf{ShamirSS}(2, 2, (\alpha_1, \alpha_2))$ *over prime field F of order p. Define $u := 2^{-i}\cdot\alpha_1$ and $v := 2^{-j}\cdot\alpha_2$.*

$$\mathsf{SD}\Big(\vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(s))\Big) = \frac{1}{2p}\cdot\left|\Sigma_{u,v}^{(\Delta_1)} - \Sigma_{u,v}^{(\Delta_2)}\right|$$

where $\Delta_1 = \alpha_2^{-1}2^{-1}\cdot(2^j - 1) - \alpha_1^{-1}2^{-1}\cdot(2^i - 1)$ and $\Delta_2 = (s\cdot 2^{-1})\cdot(\alpha_1^{-1} - \alpha_2^{-1}) + \alpha_2^{-1}2^{-1}\cdot(2^j - 1) - \alpha_1^{-1}2^{-1}\cdot(2^i - 1)$, both are automorphisms over F.

Note that $\Delta_2 = (s\cdot 2^{-1})\cdot(\alpha_1^{-1} - \alpha_2^{-1}) + \Delta_1$ and Δ_1 is a constant independent of s.

Proof. Given $s \in F_p$, by following the same derivation as in the proof of Lemma 39, we get

$$\begin{aligned}
&\mathsf{SD}\Big(\vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(s))\Big)\\
&\quad= 2\cdot\Big|\mathop{\mathbb{E}}_{x}\big[\mathbb{1}_E(2^{-i}\alpha_1 x + 2^{-i} - 1)\cdot\mathbb{1}_E(2^{-j}\alpha_2 x + 2^{-j} - 1)\big]\\
&\qquad\qquad - \mathop{\mathbb{E}}_{x}\big[\mathbb{1}_E(2^{-i}\alpha_1 x + 2^{-i}s + 2^{-i} - 1)\cdot\mathbb{1}_E(2^{-j}\alpha_2 x + 2^{-j}s + 2^{-j} - 1)\big]\Big|\\
&\qquad\qquad\qquad\text{(By Proposition 14, } E := \mathrm{PHYS}_0^{-1}(0))\\
&\quad= 2\cdot\Big|\mathop{\mathbb{E}}_{y}\big[\mathbb{1}_E(2^{-i}\alpha_1 y)\cdot\mathbb{1}_E(2^{-j}\alpha_2 y - s')\big]\\
&\qquad\qquad - \mathop{\mathbb{E}}_{x}\big[\mathbb{1}_E(2^{-i}\alpha_1 x + 2^{-i}s + 2^{-i} - 1)\cdot\mathbb{1}_E(2^{-j}\alpha_2 x + 2^{-j}s + 2^{-j} - 1)\big]\Big|\\
&\qquad\qquad\qquad\Big(\text{By substituting } y := x + \alpha_1^{-1} - 2^i\cdot\alpha_1^{-1}\\
&\qquad\qquad\qquad\qquad\text{and } s' := 2^{i-j}\alpha_1^{-1}\alpha_2\cdot(2^{-i} - 1) - 2^{-j} + 1\Big)\\
&\quad= 2\cdot\Big|\mathop{\mathbb{E}}_{y}\big[\mathbb{1}_E(2^{-i}\alpha_1 y)\cdot\mathbb{1}_E(2^{-j}\alpha_2 y + s')\big]\\
&\qquad\qquad - \mathop{\mathbb{E}}_{z}\big[\mathbb{1}_E(2^{-i}\alpha_1 z)\cdot\mathbb{1}_E(2^{-j}\alpha_2 z - s'')\big]\Big|\\
&\qquad\qquad\Big(\text{By substituting } z := x + \alpha_1^{-1}\cdot s + \alpha_1^{-1} - 2^i\cdot\alpha_1^{-1}\\
&\qquad\qquad\qquad\text{and } s'' := 2^{-j}\cdot(\alpha_1^{-1}\alpha_2 - 1)s + 2^{i-j}\alpha_1^{-1}\alpha_2\cdot(2^{-i} - 1) - 2^{-j} + 1\Big)
\end{aligned}$$

All variable substitutions $x \mapsto y$, $x \mapsto z$, and $s \mapsto s''$ made above are automorphisms over F^*. Applying Claim 22 to above, we get

$$\mathsf{SD}\Big(\vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(s))\Big)$$

$$
\begin{aligned}
&= 2 \cdot \left| \mathop{\mathbb{E}}_{y}\left[\left(\frac{1+\mathrm{sign}_p(2^{-i}\alpha_1 y \cdot 2^{-1})}{2}\right) \cdot \left(\frac{1+\mathrm{sign}_p((2^{-j}\alpha_2 y - s') \cdot 2^{-1})}{2}\right)\right] \right. \\
&\qquad \left. - \mathop{\mathbb{E}}_{z}\left[\left(\frac{1+\mathrm{sign}_p(2^{-i}\alpha_1 z \cdot 2^{-1})}{2}\right) \cdot \left(\frac{1+\mathrm{sign}_p((2^{-j}\alpha_2 z - s'') \cdot 2^{-1})}{2}\right)\right] \right| \\
&= \frac{1}{2} \cdot \left| \mathop{\mathbb{E}}_{y}\left[\mathrm{sign}_p(2^{-i}\alpha_1 y \cdot 2^{-1}) \cdot \mathrm{sign}_p((2^{-j}\alpha_2 y - s') \cdot 2^{-1})\right] \right. \\
&\qquad\qquad \left. - \mathop{\mathbb{E}}_{z}\left[\mathrm{sign}_p(2^{-i}\alpha_1 z \cdot 2^{-1}) \cdot \mathrm{sign}_p((2^{-j}\alpha_2 z - s'') \cdot 2^{-1})\right] \right| \\
&= \frac{1}{2p} \cdot \left| \sum_{X \in F} \mathrm{sign}_p(2^{-i}\alpha_1 X) \cdot \mathrm{sign}_p(2^{-j}\alpha_2 X - s' \cdot 2^{-1}) \right. \\
&\qquad\qquad \left. - \sum_{X' \in F} \mathrm{sign}_p(2^{-i}\alpha_1 X') \cdot \mathrm{sign}_p(2^{-j}\alpha_2 X' - s'' \cdot 2^{-1}) \right| \\
&= \frac{1}{2p} \cdot \left| \sum_{X \in F} \mathrm{sign}_p(2^{-i}\alpha_1 X) \cdot \mathrm{sign}_p(2^{-j}\alpha_2 \cdot (X - s' \cdot 2^{-1} \cdot 2^j \cdot \alpha_2^{-1})) \right. \\
&\qquad\qquad \left. - \sum_{X \in F} \mathrm{sign}_p(2^{-i}\alpha_1 X) \cdot \mathrm{sign}_p(2^{-j}\alpha_2 (X - s'' \cdot 2^{-1} \cdot 2^j \cdot \alpha_2^{-1})) \right| \\
&= \frac{1}{2p} \cdot \left| \Sigma^{(\Delta_1)}_{2^{-i}\alpha_1, 2^{-j}\alpha_2} - \Sigma^{(\Delta_2)}_{2^{-i}\alpha_1, 2^{-j}\alpha_2} \right|
\end{aligned}
$$

where Δ_1 and Δ_2 are:

$$
\begin{aligned}
\Delta_1 &= s' \cdot 2^{-1} \cdot 2^j \cdot \alpha_2^{-1} = \alpha_2^{-1} 2^{-1} \cdot (2^j - 1) - \alpha_1^{-1} 2^{-1} \cdot (2^i - 1) \\
\Delta_2 &= s'' \cdot 2^{-1} \cdot 2^j \cdot \alpha_2^{-1} \\
&= (s \cdot 2^{-1}) \cdot (\alpha_1^{-1} - \alpha_2^{-1}) + \alpha_2^{-1} 2^{-1} \cdot (2^j - 1) - \alpha_1^{-1} 2^{-1} \cdot (2^i - 1)
\end{aligned}
$$

◀

With this lemma, one can follow the same lines of calculations as in the proof for Theorem 7 in Section 7.1, and achieve a tight bound for $\mathsf{SD}\left(\overrightarrow{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(0))\ ,\ \overrightarrow{\mathrm{PHYS}}_{i,j}(\mathsf{Share}(s))\right)$ as the one stated in Theorem 7. This then generates a variant of Corollary 11 and Corollary 12 for physical bit leakages for Fermat primes, proving Corollary 16 for Fermat primes.

E.4 Proof of Corollary 17 and Corollary 18

Proof of Corollary 17. For the proof, fix $\alpha_1 = 1$ and $\alpha_2 = 2^{\lfloor \lambda/2 \rfloor} - 1$. We shall compute $\varepsilon_{\mathrm{LSB}}(2^i \cdot \alpha_1, \alpha_2)$ for all $i \in \{0, 1, \ldots, \lambda - 1\}$ using Theorem 7. The bound in our corollary will be the maximum of these individual upper bounds on $\varepsilon_{\mathrm{LSB}}(\cdot)$.

Case A: $i = 0$. We are interested in computing the security of the evaluation places $(2^i\alpha_1, \alpha_2)$ We use $(u, v) = (1, 2^t - 1)$, where $t = \lfloor \lambda/2 \rfloor$. Note that u, v are relatively prime and $|u|_p = 1$ and $|v|_p = 2^t - 1$. Both these evaluation places are odd. Therefore, by Theorem 7, we have

$$
\varepsilon_{\mathrm{LSB}}(2^i \cdot \alpha_1, \alpha_2) \leqslant \frac{1}{2^t - 1} + \frac{4 + 4 \cdot (2^t - 1) - 2}{p}.
$$

Case B: $1 \leqslant i \leqslant \lfloor \lambda/2 \rfloor$. We are interested in the security of $(u, v) = (2^i, 2^t - 1)$, where $t = \lfloor \lambda/2 \rfloor$. Note that u and v are relatively prime, u is even, and v is odd. Therefore, by

Theorem 7, we have

$$\varepsilon_{\mathrm{LSB}}(2^i\cdot\alpha_1,\alpha_2)\leqslant\frac{4\cdot 2^i+4\cdot(2^t-1)-2}{p}.$$

Case C: $\lfloor\lambda/2\rfloor+1\leqslant i\leqslant\lambda-1$. We are interested in the security of $(u,v)=(2^i,2^t-1)$, where $t=\lfloor\lambda/2\rfloor$. Note that $t+1\leqslant i\leqslant\lambda-1$. Define $(u',v'):=2^{\lambda-t}\cdot(u,v)\in[u\colon v]$. Observe that

$$\begin{aligned} u'&=2^{\lambda-t}\cdot u=2^{i-t} && \bmod 2^\lambda-1\\ v'&=2^{\lambda-t}\cdot v=-(2^{\lambda-t}-1) && \bmod 2^\lambda-1.\end{aligned}$$

Substitute $u'=2^j$, where $1\leqslant j\leqslant\lfloor\lambda/2\rfloor$, and $v'=-(2^{\lambda-t}-1)$. Therefore, by Theorem 7, we have

$$\varepsilon_{\mathrm{LSB}}(2^i\cdot\alpha_1,\alpha_2)\leqslant\frac{4\cdot 2^j+4\cdot(2^{\lambda-t}-1)-2}{p}.$$

Appendix E.4.1 proves the following upper bound on the insecurity for all $0\leqslant i<\lambda$.

$$\varepsilon_{\mathrm{LSB}}(2^i\cdot\alpha_1,\alpha_2)\leqslant\frac{4\cdot(2^{\lfloor\lambda/2\rfloor}+2^{\lceil\lambda/2\rceil})-6}{p}$$

and this concludes the proof. ◀

Proof of Corollary 18. Just as we did for the proof of Corollary 17 above, we begin by fixing $\alpha_1=1$ and $\alpha_2=2^{\lambda/2}-1$, and consider $\varepsilon_{\mathrm{LSB}}(2^i\cdot\alpha_1,\alpha_2)$ for all $i\in\{0,1,\ldots,\lambda\}$. Note that when $p=2^\lambda+1$ is a Fermat prime, $\lambda \mod 2=0$, i.e.$\lambda/2\in\mathbb{Z}$. As defined in the algorithm in Figure 4, set $B:=\lceil 2^{3/4}\sqrt{p}\rceil>\lceil\sqrt{p}\rceil=2^{\lambda/2}+1$.

Case A: $i=0$. Since $(2^i\alpha_1,\alpha_2)=(\alpha_1,\alpha_2)=(1,2^{\lambda/2}-1)$, we may choose $(u,v)=(1,2^{\lambda/2}-1)$ because both $|u|_p=1$ and $|v|_p=2^{\lambda/2}-1$ are less than B. Hence, the LSB advantage is, by Theorem 7,

$$\begin{aligned}\varepsilon_{\mathrm{LSB}}(2^i\cdot\alpha_1,\alpha_2)&\leqslant\frac{g^2}{|u|_p|v|_p}+\frac{4(|u|_p+|v|_p)-3/2}{p}\\ &=\frac{1}{2^{\lambda/2}-1}+\frac{4\cdot 2^{\lambda/2}-3/2}{p}=\frac{4\cdot 2^{\lambda/2}}{4\cdot(2^\lambda-2^{\lambda/2})}+\frac{4\cdot 2^{\lambda/2}-3/2}{p}\\ &\leqslant\frac{4\cdot 2^{\lambda/2}}{p}+\frac{4\cdot 2^{\lambda/2}-3/2}{p} \qquad (p=2^\lambda+1\leqslant 4(2^\lambda-2^{\lambda/2})\text{ for }\lambda\geqslant 2)\\ &=\frac{8\cdot 2^{\lambda/2}-3/2}{p}\end{aligned}$$

Case B: $i\in\{1,2,\ldots,\lambda/2\}$. We have $(2^i\alpha_1,\alpha_2)=(2^i,2^{\lambda/2}-1)$. We can still choose $u=2^i\alpha_1=2^i$ and $v=\alpha_2=2^{\lambda/2}-1$ as both $|u|_p$ and $|v|_p$ would be still less than B, except $|u|_p$ would be even this time. Then, again by Theorem 7,

$$\begin{aligned}\varepsilon_{\mathrm{LSB}}(2^i\cdot\alpha_1,\alpha_2)&\leqslant\frac{4(|u|_p+|v|_p)-3/2}{p}=\frac{4\cdot 2^i+4\cdot 2^{\lambda/2}-11/2}{p}\\ &\leqslant\frac{4\cdot 2^{\lambda/2}+4\cdot 2^{\lambda/2}-11/2}{p}\\ &=\frac{8\cdot 2^{\lambda/2}-11/2}{p}\end{aligned}$$

Case C: $i \in \{\lambda/2+1, \lambda/2+2, \ldots, \lambda\}$**.** For this case, $2^i > \sqrt{p}$ and hence we need to choose u and v differently. For simplicity, denote $i = \lambda/2 + j$ for $j \in \{1, 2, \ldots, \lambda/2\}$. Let $u = 2^{\lambda/2}2^i$ and $v = 2^{\lambda/2}(2^{\lambda/2}-1)$. Clearly $(u, v) \in [\alpha_1 \colon \alpha_2]$ and they can be simplified as follows

$$\begin{aligned} u &= 2^{\lambda/2}2^i = 2^{\lambda}2^j = -2^j && \mod p \\ v &= 2^{\lambda/2}(2^{\lambda/2}-1) = 2^{\lambda} - 2^{\lambda/2} = -(2^{\lambda/2}+1) && \mod p \end{aligned}$$

Then $|u|_p = 2^j < \sqrt{p} < B$ and $|v|_p = 2^{\lambda/2} + 1 = \lceil \sqrt{p} \rceil < B$. Hence, by Theorem 7,

$$\begin{aligned} \varepsilon_{\text{LSB}}(2^i \cdot \alpha_1, \alpha_2) &\leqslant \frac{4(|u|_p + |v|_p) - 3/2}{p} = \frac{4 \cdot 2^j + 4 \cdot 2^{\lambda/2} + 4 - 3/2}{p} \\ &\leqslant \frac{8 \cdot 2^{\lambda/2} - 5/2}{p}. \end{aligned}$$

This completes the proof. ◀

E.4.1 Proof of inequality used in the proof of Corollary 17

Observe that $\lambda - t = \lambda - \lfloor \lambda/2 \rfloor = \lceil \lambda/2 \rceil \geqslant \lfloor \lambda/2 \rfloor = t$. Therefore, for $1 \leqslant i \leqslant \lambda - 1$, we have

$$\varepsilon_{\text{LSB}}(2^i \cdot \alpha_1, \alpha_2) \leqslant \frac{4 \cdot 2^t + 4 \cdot (2^{\lambda-t} - 1) - 2}{p}.$$

All that remains is to prove that this upper bound also holds for $\varepsilon_{\text{LSB}}(2^0 \cdot \alpha_1, \alpha_2)$.

For $\lambda = 2$, we have $t = 1$. In this case, one can verify that the upper bound holds.

$$\varepsilon(2^0 \cdot \alpha_1, \alpha_2) \leqslant \frac{4 \cdot 2^t + 4 \cdot (2^{\lambda-t} - 1) - 2}{p}.$$

For $\lambda \geqslant 3$, note that if $p = 2^{\lambda} - 1$ is a Mersenne prime, then λ must be odd. Therefore, we have $\lambda - t = t + 1$ and $p = 2^{2t+1} - 1$. Therefore, we need to prove that

$$\varepsilon(2^0 \cdot \alpha_1, \alpha_2) = \frac{1}{2^t - 1} + \frac{4 + 4 \cdot (2^t - 1) - 2}{p} \leqslant \frac{4 \cdot 2^t + 4 \cdot (2^{t+1} - 1) - 2}{p}.$$

This bound is equivalent to proving

$$\begin{aligned} & & \frac{1}{2^t - 1} &\leqslant \frac{4 \cdot (2^{t+1} - 1)}{2^{2t+1} - 1} & \\ &\iff & \frac{1}{T - 1} &\leqslant \frac{4 \cdot (2T - 1)}{2T^2 - 1} & \text{(Substitute } T = 2^t\text{)} \\ &\iff & 0 &\leqslant 6T^2 - 12T + 5 & \\ &\iff & 1/6 &\leqslant (T - 1)^2, & \end{aligned}$$

which is true for all $t \geqslant 1$. Therefore, the overall maximum is

$$\frac{4 \cdot 2^{\lfloor \lambda/2 \rfloor} + 4 \cdot (2^{\lceil \lambda/2 \rceil} - 1) - 2}{p} = \frac{4 \cdot (2^{\lfloor \lambda/2 \rfloor} + 2^{\lceil \lambda/2 \rceil}) - 6}{p}.$$

◀

F Extension to arbitrary Number of Parties and proof

F.1 Proof of Corollary 20

Proof of Corollary 20. Choose arbitrary distinct evaluation places $\alpha_1, \alpha_2, \alpha_4, \ldots, \alpha_n \in F_p^*$. Choose α_3 uniformly at random from the set $F_p \setminus \{\alpha_1, \alpha_2, \alpha_4, \ldots, \alpha_n\}$.

Define $\beta_i := \left(\alpha_i \prod_{j \neq i} (\alpha_i - \alpha_j)\right)^{-1}$ as in Theorem 19, for $i \in \{1, \ldots, n\}$. Observe that choosing $\vec{\alpha}$ at random does not necessarily imply that $\vec{\beta}$ is uniformly and independently random over F_p. For this paper, we will prove a result that is easy to prove and sufficient for our context.

Define

$$\gamma_1 := \alpha_1 \prod_{j \neq 1} (\alpha_1 - \alpha_j), \quad \text{and } \gamma_2 := \alpha_2 \prod_{j \neq 2} (\alpha_2 - \alpha_j).$$

Then, we have

$$\begin{aligned}
[\gamma_1 : \gamma_2] &= \left[\alpha_1 \prod_{j \neq 1} (\alpha_1 - \alpha_j) \ : \ \alpha_2 \prod_{j \neq 2} (\alpha_2 - \alpha_j)\right] && \text{(By definition)} \\
&= \left[1 : \ -\frac{\alpha_2}{\alpha_1} \cdot \prod_{j \geqslant 3} \left(\frac{\alpha_2 - \alpha_j}{\alpha_1 - \alpha_j}\right)\right] && \text{(Because } \alpha_1 \neq 0 \text{ and } \alpha_1 \notin \{\alpha_3, \alpha_4, \ldots, \alpha_n\}\text{)} \\
&= \left[1 \colon \Delta \cdot \left(\frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}\right)\right], && \text{where } \Delta := -\frac{\alpha_2}{\alpha_1} \cdot \prod_{j \geqslant 4} \left(\frac{\alpha_2 - \alpha_j}{\alpha_1 - \alpha_j}\right) \\
&= \left[1 \colon \underbrace{\Delta \cdot \left(1 + \frac{\alpha_2 - \alpha_1}{\alpha_1 - \alpha_3}\right)}_{=:\Gamma}\right]
\end{aligned}$$

We make the following observations.

1. $\Delta \neq 0$, because $\alpha_2 \neq 0$ and $\alpha_2 \notin \{\alpha_4, \ldots, \alpha_n\}$.
2. $\left|\left\{\Delta \cdot \left(1 + \frac{\alpha_2 - \alpha_1}{\alpha_1 - \alpha_3}\right) \colon \alpha_3 \in F_p^* \setminus \{\alpha_1, \alpha_2, \alpha_4, \ldots, \alpha_n\}\right\}\right| = (p-1) - (n-1) = p - n$, since the mapping $\alpha_3 \mapsto \Gamma$ is a bijection.

Note that $[\beta_1 : \beta_2]$ is identical to $[\gamma_1^{-1} : \gamma_2^{-1}] = \left[1 : \Gamma^{-1}\right]$. By Corollary 16, there are at most

$$p \cdot \left(\frac{1}{4 \ln 2} \cdot \frac{(\ln p)^2}{\sqrt{p}} + \frac{5}{2 \ln 2} \cdot \frac{\ln p}{\sqrt{p}}\right) = \frac{1}{4 \ln 2} \cdot (\ln p)^2 \sqrt{p} + \frac{5}{2 \ln 2} \cdot (\ln p) \sqrt{p}$$

values of $\alpha_3 \in F_p^* \setminus \{\alpha_1\}$ such that the algorithm in Figure 3 returns "may be insecure" for $\mathsf{ShamirSS}(2, 2, (\beta_1, \beta_2))$. Thus, there are at most

$$\frac{1}{4 \ln 2} \cdot (\ln p)^2 \sqrt{p} + \frac{5}{2 \ln 2} \cdot (\ln p) \sqrt{p}$$

values of $\alpha_3 \in F_p^* \setminus \{\alpha_1, \alpha_2, \alpha_4, \ldots, \alpha_n\}$ that are classified as "may be insecure". This implies that, when $\alpha_3 \leftarrow F_p^* \setminus \{\alpha_1, \alpha_2, \alpha_4, \ldots, \alpha_n\}$, the probability the algorithm reports "may be insecure" is at most

$$\left(\frac{1}{4 \ln 2} \cdot (\ln p)^2 \sqrt{p} + \frac{5}{2 \ln 2} \cdot (\ln p) \sqrt{p}\right) / (p - n).$$

This completes the proof. ◀

F.2 Proof of Theorem 19

Before we dive into the proof of Theorem 19, we shall recall some definitions and basic results regarding generalized Reed-Solomon (GRS) code and Fourier analysis of Boolean functions that are necessary to understand the proof of Theorem 19.

F.2.1 Generalized Reed-Solomon Code

A generalized Reed-Solomon code over a prime field F with message length k and block length n consists of an encoding function $\mathsf{Enc}\colon F^k \to F^n$ and decoding function $\mathsf{Dec}\colon F^n \to F^k$. It is specified by distinct *evaluation places* $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ and a *scaling vector* $\vec{u}$ such that for all $1 \leqslant i \leqslant n$, $u_i \in F^*$. Given $\vec{\alpha}$ and $\vec{u}$, the *encoding function* is

$$\mathsf{Enc}(m_1, \dots, m_k) := (u_1 \cdot f(\alpha_1), \dots, u_n \cdot f(\alpha_n)),$$

where $f(X) := m_1 + m_2 X + \cdots + m_k X^{k-1}$. We represent this code as $[n, k, \vec{\alpha}, \vec{u}]_F$-GRS.

The following standard properties of generalized Reed-Solomon codes shall be helpful for our extension to an arbitrary number of parties [28, 40].

▶ **Theorem 43** (Properties of GRS). *The dual code of $[n, k, \vec{\alpha}, \vec{u}]_F$-GRS is identical to the $[n, n-k, \vec{\alpha}, \vec{v}]_F$-GRS, where for all $1 \leqslant i \leqslant n$,*

$$v_i^{-1} := u_i \prod_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_i - \alpha_j).$$

In particular, when $k = n - 1$, the dual code is the set $\{\beta \cdot (v_1, v_2, \dots, v_n) \colon \beta \in F\}$, a dimension one vector space over F.

We will apply this theorem to the dual of the code containing all possible secret shares of the secret 0 in $[n, n-1, \vec{\alpha}]$-Shamir secret-sharing.

F.2.2 Fourier Analysis Basics

We use Fourier analysis on prime field F of order p. Define $\omega := \exp(2\pi \mathrm{i}/p)$. For any functions $f, g\colon F \to \mathbb{C}$, we define the inner product as

$$\langle f, g \rangle := \frac{1}{p} \sum_{x \in F} f(x) \cdot \overline{g(x)},$$

where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$. For $z \in \mathbb{C}$, $|z| := \sqrt{z\overline{z}}$. For any $\alpha \in F$, define the function $\widehat{f}\colon F \to \mathbb{C}$ as follows.

$$\widehat{f}(\alpha) := \frac{1}{p} \sum_{x \in F} f(x) \cdot \omega^{-\alpha x}.$$

The Fourier transform maps the function f to the function $\widehat{f}$.

▶ **Lemma 44** (Fourier Inversion Formula). $f(x) = \sum_{\alpha \in F} \widehat{f}(\alpha) \cdot \omega^{\alpha x}$.

The following propositions will be useful, which follow directly from the definition.

▶ **Proposition 45.** *Let $S, T \subseteq F$ be a partition of F. For all $\alpha \in F$,*

$$\widehat{\mathbb{1}_S}(\alpha) = -\widehat{\mathbb{1}_T}(\alpha).$$

▶ **Proposition 46** (Properties of Fourier Coefficients)**.** *For all $S \subseteq F$ and $x, \alpha \in F$, it holds that*

$$\widehat{\mathbb{1}_{x+S}}(\alpha) = \widehat{\mathbb{1}_S}(\alpha) \cdot \omega^{-\alpha \cdot x},$$
$$\widehat{\mathbb{1}_S}(x \cdot \alpha) = \widehat{\mathbb{1}_{S \cdot x}}(\alpha).$$

F.2.3 Some Preparatory Results

The following result rewrites the statistical distance between two leakage distributions using the Fourier coefficients of appropriate indicator functions.

▶ **Proposition 47.** *Consider* $\mathsf{ShamirSS}(n, n)$ *over a prime field F. Let $C_0^{\perp}$ be the dual code of* $\mathsf{Share}(0)$*. For any one-bit leakage function, $\vec{\tau} \colon F^n \to \{0,1\}^n$, the following identity holds for any secret $s \in F$.*

$$\begin{aligned} &2\mathsf{SD}(\vec{\tau}(\mathsf{Share}(0))\ ,\ \vec{\tau}(\mathsf{Share}(s))) \\ &\quad = 2^n \left| \sum_{\vec{\gamma} \in \mathcal{C}_0^{\perp} \setminus \vec{0}} \left(\prod_{i=1}^{n} \widehat{\mathbb{1}}_{\tau_i^{-1}(0)}(\gamma_i) \right) \cdot \left(1 - \omega^{s \cdot (\gamma_1 + \cdots + \gamma_n)} \right) \right|. \end{aligned}$$

Proof of Proposition 47. The following identity is known in the literature (see [41] for proof).

$$\begin{aligned} &2\mathsf{SD}(\vec{\tau}(\mathsf{Share}(0))\ ,\ \vec{\tau}(\mathsf{Share}(s))) \\ &\quad = \sum_{\vec{\ell} \in \{0,1\}^n} \left| \sum_{\vec{\gamma} \in \mathcal{C}_0^{\perp}} \left(\prod_{i=1}^{n} \widehat{\mathbb{1}_{\tau_i^{-1}(\ell_i)}}(\gamma_i) \right) \cdot \left(1 - \omega^{s \cdot (\gamma_1 + \cdots + \gamma_n)} \right) \right| \end{aligned}$$

By Proposition 45, $\widehat{\mathbb{1}_{\tau_i^{-1}(\ell_i)}}(\gamma_i) = \widehat{\mathbb{1}_{\tau_i^{-1}(1-\ell_i)}}(\gamma_i)$ since $\tau_i^{-1}(\ell_i)$ and $\tau_i^{-1}(1-\ell_i)$ are a partition of F. Using this property, one can verify for every $\vec{\ell}, \vec{\ell'} \in \{0,1\}^n$, it holds that

$$\begin{aligned} &\left| \sum_{\vec{\gamma} \in \mathcal{C}_0^{\perp}} \left(\prod_{i=1}^{n} \widehat{\mathbb{1}_{\tau_i^{-1}(\ell_i)}}(\gamma_i) \right) \cdot \left(1 - \omega^{s \cdot (\gamma_1 + \cdots + \gamma_n)} \right) \right| \\ &\quad = \left| \sum_{\vec{\gamma} \in \mathcal{C}_0^{\perp}} \left(\prod_{i=1}^{n} \widehat{\mathbb{1}_{\tau_i^{-1}(\ell'_i)}}(\gamma_i) \right) \cdot \left(1 - \omega^{s \cdot (\gamma_1 + \cdots + \gamma_n)} \right) \right|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &2\mathsf{SD}(\vec{\tau}(\mathsf{Share}(0))\ ,\ \vec{\tau}(\mathsf{Share}(s))) \\ &\quad = 2^n \left| \sum_{\vec{\gamma} \in \mathcal{C}_0^{\perp} \setminus \vec{0}} \left(\prod_{i=1}^{n} \widehat{\mathbb{1}}_{\tau_i^{-1}(0)}(\gamma_i) \right) \cdot \left(1 - \omega^{s \cdot (\gamma_1 + \cdots + \gamma_n)} \right) \right|, \end{aligned}$$

as desired. ◀

▶ **Proposition 48.** *Let $A_1, A_2, \ldots, A_n \subseteq F$ and $\beta_1, \beta_2, \ldots, \beta_n \in F^*$. Then, for any $s \in F$, the following identity holds.*

$$\sum_{t \in F} \prod_{i=1}^{n} \left(\widehat{\mathbb{1}_{A_i \cdot \beta_i}}(t) \cdot \omega^{s \cdot t \cdot \beta_i} \right)$$

$$= \frac{1}{p^{n-1}} \sum_{\substack{x_n \in A_n \cdot \beta_n \\ \vdots \\ x_3 \in A_3 \cdot \beta_3}} \mathsf{card}(A_2) - \mathsf{card}\left(\left(A_2 \cdot \beta_2 + \sum_{i=3}^{n} x_i - s \cdot \sum_{i=1}^{n} \beta_i\right) \bigcap A_1 \cdot \beta_1\right)$$

Proof of Proposition 48. We shall extensively use the linear property of Fourier coefficients.

$$\begin{aligned}
&\sum_{t \in F} \prod_{i=1}^{n} \left(\widehat{\mathbb{1}_{A_i \cdot \beta_i}}(t) \cdot \omega^{s \cdot t \cdot \beta_i}\right) \\
&= \sum_{t \in F} \prod_{i} \left(\frac{1}{p} \sum_{x_i \in F} \mathbb{1}_{A_i \cdot \beta_i}(x_i) \cdot \omega^{-t \cdot x_i} \cdot \omega^{s \cdot t \cdot \beta_i}\right) && \text{(Fourier expansion)} \\
&= \frac{1}{p^n} \sum_{t \in F} \sum_{\vec{x} \in F^n} \left(\prod_{i=1}^{n} \mathbb{1}_{A_i \cdot \beta_i}(x_i) \cdot \omega^{-t \cdot x_i} \cdot \omega^{s \cdot t \cdot \beta_i}\right) && \text{(Linearity)} \\
&= \frac{1}{p^n} \sum_{\vec{x} \in F^n} \left(\prod_{i=1}^{n} \mathbb{1}_{A_i \cdot \beta_i}(x_i)\right) \sum_{t \in F} \omega^{-t \cdot (x_1 + \cdots + x_n - s \cdot (\beta_1 + \cdots + \beta_n))} && \text{(Linearity)} \\
&= \frac{1}{p^{n-1}} \sum_{\substack{\vec{x} \in F^n : \\ x_1 + \cdots + x_n = s \cdot (\beta_1 + \cdots + \beta_n)}} \left(\prod_{i=1}^{n} \mathbb{1}_{A_i \cdot \beta_i}(x_i)\right) && \text{(Sum of roots of unity)}
\end{aligned}$$

Now, replacing $x_1 = s \cdot (\beta_1 + \cdots + \beta_n) - (x_2 + \cdots + x_n)$ yields

$$\begin{aligned}
&\frac{1}{p^{n-1}} \sum_{x_2, \ldots, x_n \in F} \mathbb{1}_{A_1 \cdot \beta_1}(s \cdot (\beta_1 + \cdots + \beta_n) - (x_2 + \cdots + x_n)) \cdot \prod_{i=2}^{n} \mathbb{1}_{A_i \cdot \beta_i}(x_i) \\
&= \frac{1}{p^{n-1}} \sum_{\substack{x_n \in A_n \cdot \beta_n \\ \vdots \\ x_3 \in A_3 \cdot \beta_3}} \sum_{x_2 \in F} \mathbb{1}_{A_2 \cdot \beta_2}(x_2) \cdot \mathbb{1}_{A_1 \cdot \beta_1}(s \cdot (\beta_1 + \cdots + \beta_n) - (x_2 + \cdots + x_n))
\end{aligned}$$

Let us take a detour and simplify the inner summand using linear properties of sets and indicator functions as follows.

$$\begin{aligned}
&\sum_{x_2 \in F} \mathbb{1}_{A_2 \cdot \beta_2}(x_2) \cdot \mathbb{1}_{A_1 \cdot \beta_1}(s \cdot (\beta_1 + \cdots + \beta_n) - (x_2 + \cdots + x_n)) \\
&= \sum_{x_2 \in F} \mathbb{1}_{A_2 \cdot \beta_2}(x_2) \cdot \mathbb{1}_{A_1 \cdot \beta_1 - s \cdot (\beta_1 + \cdots + \beta_n) + (x_3 + \cdots + x_n)}(-x_2) \\
&= \sum_{x_2 \in F} \mathbb{1}_{A_2 \cdot \beta_2}(x_2) \cdot \mathbb{1}_{-A_1 \cdot \beta_1 + s \cdot (\beta_1 + \cdots + \beta_n) - (x_3 + \ldots + x_n)}(x_2) \\
&= \mathsf{card}(A_2 \cdot \beta_2 \cap (-A_1 \cdot \beta_1 - (x_3 + \ldots + x_n) + s \cdot (\beta_1 + \cdots + \beta_n))) \\
&= \mathsf{card}(A_2 \cdot \beta_2 + (x_3 + \cdots + x_n) - s \cdot (\beta_1 + \cdots + \beta_n) \cap (-A_1 \cdot \beta_1)) \\
&= \mathsf{card}(A_2 \cdot \beta_2) - \mathsf{card}(A_2 \cdot \beta_2 + (x_3 + \cdots + x_n) - s \cdot (\beta_1 + \cdots + \beta_n) \cap A_1 \cdot \beta_1) \\
&= \mathsf{card}(A_2) - \mathsf{card}(A_2 \cdot \beta_2 + (x_3 + \cdots + x_n) - s \cdot (\beta_1 + \cdots + \beta_n) \cap A_1 \cdot \beta_1),
\end{aligned}$$

which completes the proof. ◀

F.2.4 Putting things together and proving Theorem 19

We can now prove Theorem 19.

Proof of Theorem 19. We begin with some notations. Let $\vec{\tau}\colon F^n \to \{0,1\}^n$ be any one-bit physical leakage. Let $A_i = \tau_i^{-1}(0)$ for $1 \leqslant i \leqslant n$. By Imported Theorem 43, the dual code $C_0^{\perp}$ is the set $\{t \cdot (\beta_1, \beta_2, \dots, \beta_n) \colon t \in F\}$, where

$$\beta_i = \left(\alpha_i \prod_{j \neq i} (\alpha_i - \alpha_j)\right)^{-1}, \text{ for every } i \in \{1, 2, \dots, n\}.$$

Consider the following manipulation.

$$\begin{aligned}
&2\mathsf{SD}(\vec{\tau}(\mathsf{Share}(0))\ ,\ \vec{\tau}(\mathsf{Share}(s)))\\
&= 2^n \cdot \left| \sum_{\vec{\gamma} \in \mathcal{C}_0^{\perp} \setminus \vec{0}} \prod_{i=1}^{n} \widehat{\mathbb{1}_{\tau_i^{-1}(0)}}(\gamma_i) \cdot \left(1 - \omega^{s \cdot (\gamma_1 + \dots + \gamma_n)}\right) \right| && \text{(By Proposition 47)}\\
&= 2^n \cdot \left| \sum_{t \in F^*} \prod_{i=1}^{n} \widehat{\mathbb{1}_{A_i}}(t \cdot \beta_i) \cdot \left(1 - \omega^{s \cdot t \cdot (\beta_1 + \dots + \beta_n)}\right) \right|\\
&= 2^n \cdot \left| \sum_{t \in F^*} \prod_{i=1}^{n} \widehat{\mathbb{1}_{A_i \cdot \beta_i}}(t) - \sum_{t \in F^*} \prod_{i=1}^{n} \widehat{\mathbb{1}_{A_i \cdot \beta_i}}(t) \cdot \omega^{s \cdot t \cdot \beta_i} \right|
\end{aligned}$$

For each $s \in F$ and tuple $(x_3, x_4, \dots, x_n)$ satisfying $x_i \in A_i \cdot \beta_i$ for $3 \leqslant i \leqslant n$, we define

$$\begin{aligned}
&\varphi_{s,\vec{\tau}}(x_3, x_4, \dots, x_n) :=\\
&\qquad \sum_{x_n \in A_n \cdot \beta_n} \cdots \sum_{x_3 \in A_3 \cdot \beta_3} \mathsf{card}\left(\left(A_2 \cdot \beta_2 + \sum_{i=3}^{n} x_i - s \cdot \sum_{i=1}^{n} \beta_i \right) \bigcap A_1 \cdot \beta_1 \right).
\end{aligned}$$

Then, it follows from Proposition 48 that

$$2\mathsf{SD}(\vec{\tau}(\mathsf{Share}(0))\ ,\ \vec{\tau}(\mathsf{Share}(s))) = \frac{2^{n-1}}{p^{n-1}} \cdot \left| \varphi_{0,\vec{\tau}}(x_3, \dots, x_n) - \varphi_{s,\vec{\tau}}(x_3, \dots, x_n) \right|.$$

It suffices to prove the result when $\vec{\tau} = \vec{\mathrm{LSB}}$ (the proof for arbitrary physical bit leakage is similar). In this case, note that $A_1 = A_2 = E = F^+ \cdot 2$. Therefore, we have

$$\begin{aligned}
&\mathsf{card}\left(\left(A_2 \cdot \beta_2 + \sum_{i=3}^{n} x_i - s \cdot \sum_{i=1}^{n} \beta_i \right) \bigcap A_1 \cdot \beta_1 \right)\\
&\quad = \mathsf{card}\left(\left(F^+ \cdot 2 \cdot \beta_2 + \sum_{i=3}^{n} x_i - s \cdot \sum_{i=1}^{n} \beta_i \right) \bigcap F^+ \cdot 2 \cdot \beta_1 \right)\\
&\quad = \mathsf{card}\left(\left(F^+ \cdot \beta_2 + 2^{-1} \cdot \left(\sum_{i=3}^{n} x_i - s \cdot \sum_{i=1}^{n} \beta_i \right) \right) \bigcap F^+ \cdot \beta_1 \right)\\
&\quad = \Sigma_{\beta_1^{-1}, \beta_2^{-1}}^{\left(\Delta_{x_3, \dots, x_n}^{(s)}\right)},
\end{aligned}$$

where $\Delta_{x_3, \dots, x_n}^{(s)} := 2^{-1} \cdot (\sum_{i=3}^{n} x_i - s \cdot \sum_{i=1}^{n} \beta_i)$. Similar to the proof of Lemma 8 in Appendix C.1, we have

$$\begin{aligned}
&2\mathsf{SD}\left(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\right)\\
&\quad = \frac{2^{n-2}}{p^{n-1}} \cdot \left| \sum_{x_n \in E \cdot \beta_n} \cdots \sum_{x_3 \in E \cdot \beta_3} \left(\Sigma_{\beta_1^{-1}, \beta_2^{-1}}^{\left(\Delta_{x_3, \dots, x_n}^{(0)}\right)} - \Sigma_{\beta_1^{-1}, \beta_2^{-1}}^{\left(\Delta_{x_3, \dots, x_n}^{(s)}\right)} \right) \right|
\end{aligned}$$

$$\leqslant \frac{2^{n-2}}{p^{n-1}} \cdot \sum_{x_n \in E\cdot\beta_n} \cdots \sum_{x_3 \in E\cdot\beta_3} \left| \left(\Sigma^{\left(\Delta^{(0)}_{x_3,\dots,x_n}\right)}_{\beta_1^{-1},\beta_2^{-1}} - \Sigma^{\left(\Delta^{(s)}_{x_3,\dots,x_n}\right)}_{\beta_1^{-1},\beta_2^{-1}} \right) \right| \qquad \text{(By triangle inequality)}$$

Suppose $\mathsf{ShamirSS}(2,2,(\beta_1,\beta_2))$ have ε insecurity against LSB. Then, it follows from Lemma 8 that

$$\left| \Sigma^{\left(\Delta^{(0)}_{x_3,\dots,x_n}\right)}_{\beta_1^{-1},\beta_2^{-1}} - \Sigma^{\left(\Delta^{(s)}_{x_3,\dots,x_n}\right)}_{\beta_1^{-1},\beta_2^{-1}} \right| \leqslant 2\varepsilon p. \tag{17}$$

Applying the above inequality for every term under the summand yields:

$$\begin{aligned} 2\mathsf{SD}\Big(\vec{\mathrm{LSB}}(\mathsf{Share}(0))\ ,\ \vec{\mathrm{LSB}}(\mathsf{Share}(s))\Big) &\leqslant \frac{2^{n-2}}{p^{n-1}} \cdot \sum_{x_n \in E\cdot\beta_n} \cdots \sum_{x_3 \in E\cdot\beta_3} 2\varepsilon p \\ &\leqslant \frac{2^{n-2}}{p^{n-1}} \cdot \underbrace{(p/2)\cdots(p/2)}_{(n-2)\text{-times}} \cdot 2\varepsilon p \\ &= 2\varepsilon, \end{aligned}$$

which completes the proof. ◀

G Attack on $\mathsf{ShamirSS}(3,3,\vec{\alpha})$

Consider $\mathsf{ShamirSS}(3,3,\vec{\alpha})$ and the underlying prime field F of order $p = 4w^2+6w+9$ where $w \geqslant 4$ and $w \neq 0 \mod 3$. The evaluation places are $\vec{\alpha} = (1,\sigma,\sigma^2)$ for $\sigma = 2w\cdot 3^{-1} \in F_p$.

▷ Claim 49. $\left(2w\cdot 3^{-1}\right)^3 = 1 \mod p$ when $p = w^2+6w+9$ and $w \geqslant 4$.

Proof. $\left(2w\cdot 3^{-1}\right)^3 = 1 \mod p \iff (2w)^3 - 3^3 = 0 \mod p \iff (2w-3)\cdot(4w^2+6w+9) = 0 \mod p$ holds since $p = 4w^2+6w+9$. ◀

Observe that $2w < \sqrt{p}$ and $3 < \sqrt{p}$. Then, by our classifier in Figure 1, $[1:\sigma]$ is a good evaluation place since $[1:\sigma] = [3:2w]$, $\gcd(3,2w) = 1$, and $3\cdot 2w$ is an even integer.

Note that $1+\sigma+\sigma^2 = 1$; therefore, this secret sharing inherits the vulnerability of the additive secret sharing against LSB leakge [41]. Therefore, $\mathsf{ShamirSS}(3,3,\vec{\alpha})$ is insecure against LSB leakage, where its insecurity is $\geqslant (2/\pi)^3 \geqslant 0.25$ [43, 19].

H Example of Secure Evaluation places against Physical Bit Leakage

We consider $\mathsf{ShamirSS}(n=2,k=2,(\alpha_1,\alpha_2))$ over the prime field F of order $p = 2^\lambda - 1$ – a Mersenne prime. We deduced earlier that the security of (α_1,α_2) is identical to the security of all (u,v) in the equivalence class $[\alpha_1:\alpha_2]$. Note that $[\alpha_1:\alpha_2]$ is identical to the equivalence class $[1:\alpha]$, where $\alpha = \alpha_2\alpha_1^{-1}$. The equivalence class $[1:\alpha]$ is secure if and only if all the following equivalence classes

$$\Big\{[1:\alpha],[1:2^1\cdot\alpha],[1:2^2\cdot\alpha],\dots,[1:2^{\lambda-1}\cdot\alpha]\Big\}$$

are secure against the PHYS leakage.

The elements generated by 2, $\langle 2\rangle = \{1,2,2^2,\dots,2^{\lambda-1}\}$, is a cyclic subgroup of F^*. Let $\alpha\cdot\langle 2\rangle$ denote the coset $\{\alpha, 2\cdot\alpha,\dots,2^{\lambda-1}\cdot\alpha\} \in F^*/\langle 2\rangle$. Furthermore, the equivalence class $[1:\alpha]$ is secure against arbitrary physical bit leakage if (and only if) the equivalence classes $[1:\alpha']$ are secure against arbitrary physical bit leakage, for all $\alpha' \in \alpha\cdot\langle 2\rangle$.

So, in the table below, when we mention α, it implies that any $(\alpha_1, \alpha_2) \in [1 : \alpha']$ is secure against physical bit leakage attacks, where $\alpha' \in \alpha\langle 2\rangle$.

▶ Remark 50 (Adversarial LLL: A worst-case analysis). For one (α_1, α_2), there may be multiple $(u, v) \in [\alpha_1, \alpha_2]$ that the LLL algorithm can output. The output of the LLL algorithm is crucial in assessing whether evaluation places are secure. The LLL output can change our algorithm's output in Figure 2 from "secure" to "may be insecure."

For example, consider the prime $p = 127$ and $(\alpha_1, \alpha_2) = (1, 23)$. In this case, $B = \lceil 2^{3/4}\sqrt{p}\rceil = 19$. Note that $(-11, 1) \in [\alpha_1 : \alpha_2]$ and $(6, 11) \in [\alpha_1 : \alpha_2]$. If the LLL algorithm returns $(11, -1)$, our algorithm will declare "may be insecure." If the LLL algorithm returns $(6, 11)$, our algorithm will declare "secure."

Consider an "adversarial LLL" algorithm implementation for the worst-case evaluation. On input (α_1, α_2), if there is $(u, v) \in [\alpha_1 : \alpha_2]$ that makes our algorithm in Figure 2 output "may be insecure," the adversarial LLL outputs that (u, v).

For example, the element "95" in Table 1 represents the following. Any $(\alpha_1, \alpha_2) \in [1 : \alpha']$ is secure against physical bit leakage attacks, where $\alpha' \in 95\langle 2\rangle$. Note that

$$\begin{aligned}95 \cdot \langle 2\rangle =&\{95, 2 \cdot 95, 2^2 \cdot 95, \ldots, 2^{12} \cdot 95\}\\ =&\{95, 190, 380, 760, 1520, 3040, 6080, 3969, 7938, 7685, 7179, 6167, 4143\}\end{aligned}$$

Corollary 17 presents explicit evaluation places $(\alpha_1, \alpha_2) \in \left[1\colon 2^{\lfloor\lambda/2\rfloor} - 1\right]$ such that for security parameter λ,

$$\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \leqslant \frac{4 \cdot \left(2^{\lfloor\lambda/2\rfloor} + 2^{\lceil\lambda/2\rceil}\right) - 6}{p}.$$

When $\lambda = 13$ and $p = 2^{13} - 1$, it implies that $[1 : 63]$ would have $\varepsilon_{\mathrm{PHYS}}(\vec{\alpha}) \lessapprox 0.093$. However, $63 \cdot \langle 2\rangle$ is not listed in Table 1 because the "adversarial LLL" algorithm may pick $(u, v) = (1, 63)$ which is characterized as "may be insecure" by our algorithm in Figure 2.

95	97	99	101	103	107	111	113	119	121	123
125	131	133	135	137	139	143	145	147	151	153
155	157	159	161	163	165	169	173	175	179	181
183	185	187	191	197	201	203	207	209	211	213
215	217	219	221	223	225	227	229	231	233	235
237	239	243	245	247	249	251	253	267	269	271
275	277	279	281	285	287	291	293	295	297	299
303	305	309	313	317	319	323	325	329	331	333
335	337	339	349	351	355	357	359	361	363	365
369	371	373	375	377	379	391	393	395	397	399
401	403	405	407	411	413	415	419	423	427	429
433	435	437	441	443	445	447	453	457	459	461
465	467	469	471	473	475	477	487	491	493	495
497	499	501	503	505	549	551	553	555	557	559
563	567	569	573	575	581	583	587	589	591	595
599	601	603	607	611	613	615	617	619	621	623
629	633	637	651	653	655	661	667	669	671	675
677	679	687	693	695	697	699	701	713	715	717
719	725	727	729	731	735	739	743	747	751	755
757	759	761	763	795	797	799	805	807	811	813
815	821	823	825	829	843	845	847	855	857	859
863	869	871	873	875	877	879	883	885	887	889
891	893	915	917	921	923	925	927	933	937	939
943	947	949	951	953	955	957	959	971	973	975
979	987	989	991	997	1001	1005	1007	1011	1175	1181
1183	1191	1197	1199	1205	1207	1211	1213	1227	1231	1235
1237	1239	1245	1247	1253	1255	1259	1261	1263	1267	1275
1323	1327	1333	1335	1339	1341	1343	1355	1357	1359	1371
1373	1375	1387	1389	1395	1397	1403	1405	1431	1435	1439
1447	1451	1461	1467	1469	1485	1487	1491	1495	1499	1501
1503	1511	1515	1519	1525	1655	1661	1691	1693	1695	1703
1709	1711	1717	1723	1725	1727	1743	1751	1757	1759	1773
1775	1783	1787	1851	1853	1855	1871	1879	1885	1887	1899
1901	1903	1909	1915	1963	1965	1967	1973	1975	1979	1981
1983	2007	2011	2013	2015	2775	2783	2795	2799	2807	2911
2927	2935	2939	2991	2999	3003	3035	3039	3055	3551	3575

Table 1 Secure Evaluation Places against Physical Bit Leakage when $p = 2^{13} - 1$. If an element $\alpha \in F$ appears in the list above, it implies the following. Any evaluation places $(\alpha_1, \alpha_2) \in [1 : \alpha']$, where $\alpha' \in \alpha \cdot \langle 2 \rangle$, is secure against all physical bit leakage attacks.