Lecture 01: Mathematical Basics (Summations)

What I am Assuming

- I am assuming that you know asymptotic notations. For example, the big-O, little-O notations

Summation I

- Let us try to write a closed form expression for the following summation

$$S = \sum_{i=1}^{n} 1$$

- It is trivial to see that $S = n$

Summation II

- Now, let us try to write a closed form expression for the following summation

$$S = \sum_{i=1}^{n} i$$

- We can prove that $S = \frac{n(n+1)}{2}$
    - How do you prove this statement? (Use Induction? Use the formula for the Sum of an Arithmetic Progression?)
- Using Asymptotic Notation, we can say that $S = \frac{n^2}{2} + o(n^2)$

Summation III

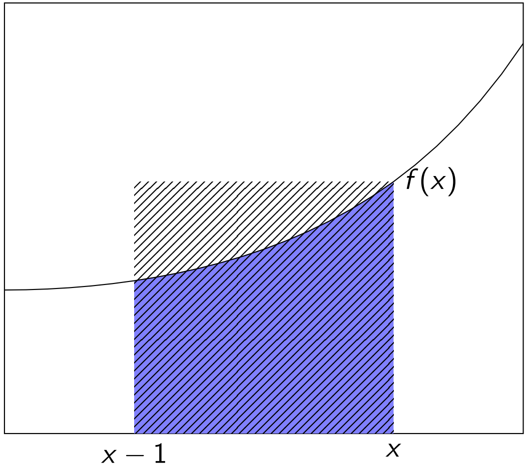
- Now, let us try to write a closed form expression for the following summation
$$S = \sum_{i=1}^{n} i^2$$
- We can prove that $S = \frac{n(n+1)(2n+1)}{6}$
  - Why is the expression on the right an integer? (Prove by induction that 6 divides $n(n+1)(2n+1)$ for all positive integer n)
  - How do you prove this statement? (Use Induction?)
- Using Asymptotic Notation, we can say that $S = \frac{n^3}{3} + o(n^3)$

- Do we see a pattern here?
- Conjecture: For $k \geqslant 1$, we have $\sum_{i=1}^{n} i^{k-1} = \frac{n^k}{k} + o(n^k)$.
    - How do we prove this statement?

Estimating Summations by Integration I

- Let f be an increasing function
- For example, $f(x) = x^{k-1}$ is an increasing function for $k > 1$ and $x \geqslant 0$

Estimating Summations by Integration II



- Observation: "Blue area under the curve" is smaller than the "Shaded area of the rectangle"
  - Blue area under the curve is:
  $$\int_{x-1}^{x} f(t)dt$$
  - Shaded area of the rectangle is:
  $$f(x)$$
  - So, we have the inequality:
  $$\int_{x-1}^{x} f(t)\,\mathrm{d}t \leqslant f(x)$$
  - Summing both side from $x = 1$ to $x = n$, we get
  $$\sum_{x=1}^{n} \int_{x-1}^{x} f(t)\,\mathrm{d}t \leqslant \sum_{x=1}^{n} f(x)$$

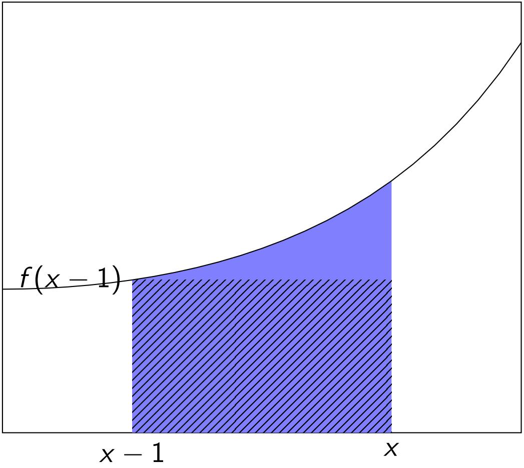
- The left-hand side of the inequality is

$$\int_0^1 f(t)\,\mathrm{d}t + \int_1^2 f(t)\,\mathrm{d}t + \cdots + \int_{n-1}^n f(t)\,\mathrm{d}t = \int_0^n f(t)\,\mathrm{d}t$$

- So, for an increasing f, we have the following lower bound.

$$\int_0^n f(t)\,\mathrm{d}t \leqslant \sum_{x=1}^{n} f(x) \qquad (1)$$

- Now, we will upper bound the summation expression. Consider the figure below



Estimating Summations by Integration VI

- Observation: "Blue area under the curve" is greater than the "Shaded area of the rectangle"
- So, we have the inequality:

$$\int_{x-1}^{x} f(t)\,\mathrm{d}t \geqslant f(x-1)$$

- Now we sum the above inequality from $x = 2$ to $x = n + 1$
- We get

$$\int_{1}^{2} f(t)\,\mathrm{d}t + \int_{2}^{3} f(t)\,\mathrm{d}t + \cdots + \int_{n}^{n+1} f(t)\,\mathrm{d}t \geqslant f(1) + f(2) + \cdots + f(n)$$

- So, for an increasing f, we get the following upper bound

$$\int_{1}^{n+1} f(t)\,\mathrm{d}t \geqslant \sum_{x=1}^{n} f(x) \tag{2}$$

Summary: Estimation of Summation using Integration

Theorem

For an increasing function f, we have

$$\int_0^n f(t)\,\mathrm{d}t \leqslant \sum_{x=1}^{n} f(x) \leqslant \int_1^{n+1} f(t)\,\mathrm{d}t$$

Exercise:

- Use this theorem to prove that $\sum_{i=1}^{n} i^{k-1} = \frac{n^k}{k} + o(n^k)$, for $k \geqslant 1$
- Consider the function $f(x) = 1/x$ to find upper and lower bounds for the sum $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ using the approach used to prove Theorem 1

Differentiation and Integration

- Differentiation: $f'(x)$ represents the slope of the curve $y = f(x)$ at x
- Integration: $\int_a^b f(t)\,\mathrm{d}t$ represents the area under the curve $y = f(x)$ between $x = a$ and $x = b$
- Increasing function:
  - Observation: The slope of an increasing function is positive
  - So, "f is increasing at x" is equivalent to "$f'(x) > 0$," i.e. f' is positive at x
- Suppose we want to mathematically write "Slope of a function f is increasing"
  - The "slope of a function f" is the function "f'"
  - So, the statement "slope of a function f is increasing" is equivalent to "$(f')' \equiv f''$ is positive"

Concave Upwards Functions

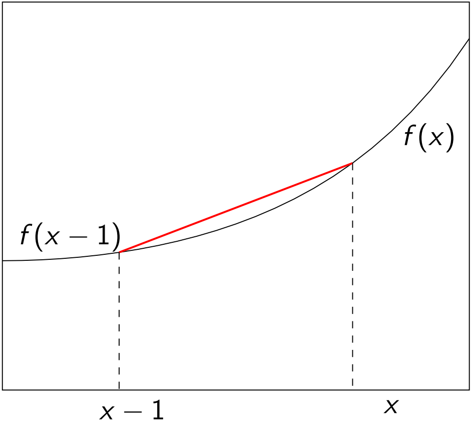
Definition (Concave Upwards Function)

A function f is *concave upwards* in the interval $[a, b]$ if f'' is positive in the interval $[a, b]$.

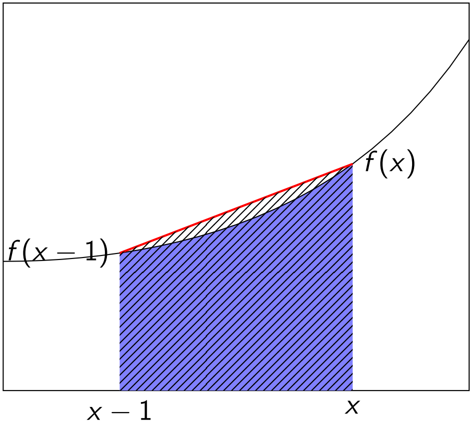
- Example of functions that concave upwards: x^2, $\exp(x)$, $1/x$ (in the interval $(0, \infty)$), $x \log x$ (in the interval $(0, \infty)$)
  - We emphasize that a "concave upwards" function need not be increasing, for example $f(x) = 1/x$ (for positive x) is decreasing

Property of Concave Upwards Function I

- Consider the coordinates $(x-1, f(x-1))$ and $(x, f(x))$
- For a concave upwards function, the secant between the two coordinates is always (on or) above the part of the curve f between the two coordinates



- So, the shaded area of the trapezium is greater than the blue area under the curve



Property of Concave Upwards Function III

- So, we get
$$\frac{f(x-1)+f(x)}{2} \geqslant \int_{x-1}^{x} f(t)\,\mathrm{d}t$$
- Now, use this new observation to obtain a better lower bound for the sum $\sum_{x=1}^{n} f(x)$
- Think: Can you get even tighter bounds?
- Additional Reading: Read on the "trapezoidal rule"

Estimating Products

- Consider the objective of estimating $n!$ using elementary functions
- Note that one can convert this estimation of products into estimation of sums by taking log. For example,

$$\ln(n!) = \sum_{i=1}^{n} \ln(i).$$

- Now, one can tightly upper and lower bound the expression $\sum_{i=1}^{n} \ln(i)$. Use the techniques in the previous slides to obtain meaningful upper and lower bounds of this expression. Suppose

$$L_n \leqslant \sum_{i=1}^{n} \ln(i) \leqslant U_n.$$

- Therefore, one concludes that

$$\exp(L_n) \leqslant n! \leqslant \exp(U_n).$$

Estimating Fractions

- Consider the objective of estimating a fraction A_n/B_n
- Suppose we have $A_n \leqslant U_n$ and $L'_n \leqslant B_n$. Note that

$$\frac{1}{B_n} \leqslant \frac{1}{L'_n}.$$

- Note that multiplying with $A_n \leqslant U_n$, one gets that

$$\frac{A_n}{B_n} \leqslant \frac{U_n}{L'_n}.$$

- To summarize, upper-bounding a fraction involves upper-bounding the numerator and lower-bounding the denominator
- Analogously, if $L_n \leqslant A_n$ and $B_n \leqslant U'_n$, then we get $\frac{L_n}{U'_n} \leqslant \frac{A_n}{B_n}$
- **Food for thought.** Provide meaningful upper and lower bound the expression $\binom{2n}{n} := \frac{(2n)!}{(n!)^2}$.

Advanced Summation Techniques

1. Abel Summation Formula:
   - Wikipedia
   - A lecture note
2. Euler-Maclaurin Summation Formula:
   - Wikipedia
   - Quick summary for applications
   - Concrete Mathematics Chapter 9.5 (you can access the book through Purdue Library)