Homework 1

1. **Estimating** $(1-x)$ **using** $\exp(\cdot)$ **function.** For $x \in [0,1)$, we know that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

   (a) **(5 points)** Prove that $1 - x \leqslant \exp\left(-x - \frac{x^2}{2}\right)$.

   Solution.

(b) **(5 points)** For $x \in [0, 1/2]$, prove that

$$1 - x \geqslant \exp\left(-x - x^2\right).$$

Solution.

2. **Tight Estimations** Provide meaningful upper and lower bounds for the following expressions.

   (a) **(5 points)** $\sum_{i=1}^{\infty}(2i-1)^{-\frac{19}{17}}$

   Note: Please evaluate/simplify the expression/bound as much as possible.

   Hint: Your upper and lower bounds should be constants.

   Solution.

(b) **(10 points)** $A_n = {}_{2n}P_n$ Hint: Note that ${}_{2n}P_n = \frac{(2n)!}{(2n-n)!}$.
Note: Please evaluate/simplify the expression/bound as much as possible.
Hint: You may want to start by upper and lower bounding $S_n = \sum_{i=1}^{n} \ln i$.
Solution.

3. **Understanding Joint Distribution.** Twelve balls are to be tossed into six bins numbered $\{1, 2, 3, 4, 5, 6\}$. Each ball is thrown into a bin uniformly and independently into the bins. For $i \in \{1, 2, 3, 4, 5, 6\}$, let X_i represent the number of balls that fall into bin i.

   (a) **(5 points)** Find the (marginal) distribution of X_5 and compute its expected value.
   Solution.

   (b) **(3 points)** Find the expected value of $X_1 + X_3 + X_5$.
   Solution.

(c) **(7 points)** Find $\mathbb{P}[X_2 = 2 | X_1 + X_3 + X_5 = 5]$.
Solution.

4. **Sending one bit.**

   Alice intends to send a bit $b \in \{0, 1\}$ to Bob. When Alice sends the bit, it goes through a series of n relays before reaching Bob. Each relay flips the received bit independently with probability p before forwarding that bit to the next relay.

   (a) **(5 points)** Show that Bob will receive the correct bit with probability

   $$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} p^{2k} \cdot (1-p)^{n-2k}.$$

   Hint: Be careful that Alice could be sending either 0 or 1.

   Solution.

(b) **(5 points)** Let us consider an alternative way to calculate this probability. We say that the relay has *bias* q if the probability it flips the bit is $(1 - q)/2$. The bias q is a real number between -1 and $+1$. Show that sending a bit through two relays with bias q_1 and q_2 is equivalent to sending a bit through a single relay with bias $q_1 \cdot q_2$.

Solution.

(c) **(5 points)** Prove that the probability you receive the correct bit when it passes through n relays is

$$\frac{1 + (1 - 2p)^n}{2}.$$

Solution.

5. **An Useful Estimate.**

For an integers n and t satisfying $0 \leqslant t \leqslant n/2$, define

$$P_n(t) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{t}{n}\right)$$

We will estimate the above expression. (*Remark*: You shall see the usefulness of this estimation in the topic "Birthday Bound" that we shall cover in the forthcoming lectures.)

(a) **(13 points)** Show that

$$\exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{6n^2}\right) \geqslant P_n(t) \geqslant \exp\left(-\frac{t^2}{2n} - \frac{t}{2n} - \frac{\Theta\left(t^3\right)}{3n^2}\right).$$

Solution.

(b) **(2 points)** Show that when $t = \sqrt{2cn}$, where c is a positive constant, the expression above is

$$P_n(t) = \exp\left(-c - \ \Theta\left(1/\sqrt{n}\right) \right).$$

Solution.

6. **(Extra Credit) An Application of Abel's Summation Formula.** One consequence of Abel's summation formula is the following summation estimate:

$$\sum_{i=1}^{n} f(i) = n \cdot f(n) - \int_{1}^{n} \lfloor t \rfloor \cdot f'(t)\ \mathrm{d}t.$$

For a constant $k \geqslant 0$, use this formula to conclude that:

$$\sum_{i=1}^{n} i^k \leqslant \frac{n^{k+1}}{k+1} + n^k - \frac{1}{k+1}.$$

Remark: Using the technique discussed in class, the upper bound we get is

$$\int_{1}^{n+1} t^k\ \mathrm{d}t \leqslant \frac{(n+1)^{k+1}}{k+1} - \frac{1}{k+1},$$

which is worse than the bound we get using Abel's formula.

Solution.

7. **(Extra Credit) An Application of Euler-Maclaurin Summation Formula.** A usable form of the Euler-Maclaurin Summation formula is the following

$$\sum_{i=1}^{n} f(i) = \int_{1}^{n} f(t)\ \mathrm{d}t + \frac{1}{2} \cdot f(n) + C_f + \sum_{1 \leqslant j \leqslant m} \frac{B_{2j}}{(2j)!} \cdot f^{(2j-1)}(n) + R_m,$$

where

(a) $m \in \{1, 2, \dots\}$ is an arbitrary parameter.

(b) C_f is a suitable constant depending only on the function f.

(c) B_{2j} is the $2j$-th Bernoulli number. For example, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, and so on...

(d) $f^{(2j-1)}$ is the $(2j-1)$-th derivative of the function f

(e) R_m is the remainder, where

$$R_m = O\left(\int_{n}^{\infty} \left| f^{(2m)}(t) \right|\ \mathrm{d}t\right).$$

Use this formula to prove the Stirling approximation: For any $m \in \{1, 2, \dots\}$

$$\ln(n!) = \sum_{i=1}^{n} \ln i = n \ln n - n + \frac{1}{2} \ln n + \frac{1}{2} \ln 2\pi + \sum_{1 \leqslant j \leqslant m} \frac{B_{2j}}{2j \cdot (2j-1)} \cdot \frac{1}{n^{2j-1}} + R_m,$$

where $R_m = O\left(\frac{(2m-2)!}{n^{2m-1}}\right)$

Solution.