Lecture 04: Groups and Fields

Group

Definition

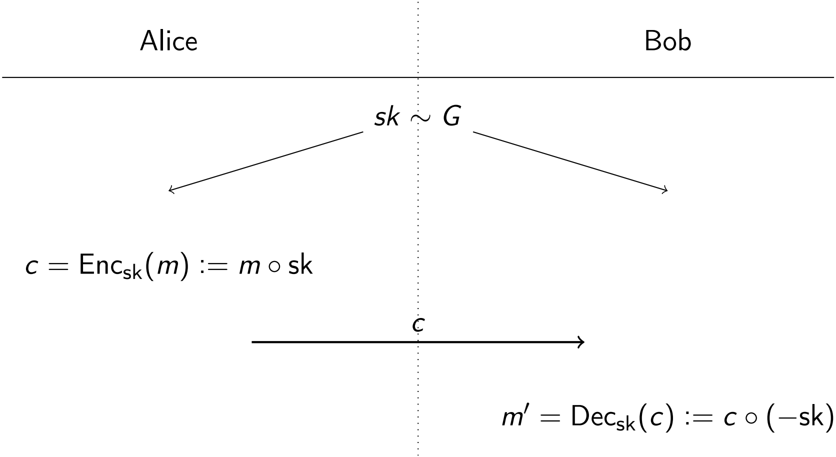
A *group*, represented by $(G, \circ)$, is defined by a set G and a binary operator $\circ$ that satisfy the following properties

1. **Closure.** For all $a, b \in G$, we have $a \circ b \in G$
2. **Associativity.** For all $a, b, c \in G$, we have $(a \circ b) \circ c = a \circ (b \circ c)$
3. **Identity.** There exists an element $e \in G$ such that for all $a \in G$, we have $a \circ e = a$
4. **Inverse.** For every element $a \in G$, there exists an element $(-a) \in G$ such that $a \circ (-a) = e$

A Quick Check

- Verify that $(\{0,1\}^n, \oplus)$, where $\oplus$ is the bit-wise XOR of bits, is a group
  - Closure and Associativity is trivial to verify
  - Show that $\overbrace{00\cdots 0}^{n\text{-times}}$ is the identity
  - Show that for $a \in \{0,1\}^n$, the inverse of a is a itself

One-time Pad extended to Arbitrary Groups



Alice

Bob

$sk \sim G$

$c = \mathrm{Enc}_{\mathrm{sk}}(m) := m \circ \mathrm{sk}$

$\xrightarrow{c}$

$m' = \mathrm{Dec}_{\mathrm{sk}}(c) := c \circ (-\mathrm{sk})$

Figure: One-time Pad encryption scheme for the group $(G, \circ)$.

Verify that the scheme is always correct

Examples I

- Groups can have infinite size. $(\mathbb{Z}, +)$, where $\mathbb{Z}$ is the set of all integers and $+$ is integer addition, is a group (Verify that it satisfies all properties of a group)
- Groups can have finite size. $(\mathbb{Z}_n, +)$, where $\mathbb{Z}_n = \{0, \ldots, n-1\}$ and $+$ is integer addition $\mod n$, is a group (Verify that it satisfies all properties of a group)

Examples II

Following are NOT groups. Find which rule is violated.

- $(\mathbb{Z}, \times)$, where $\times$ is the integer multiplication
- $(\mathbb{Z}^*, \times)$, where $\mathbb{Z}^*$ is the set of all non-zero integers and $\times$ is the integer multiplication
- $(\mathbb{Q}, \times)$, where $\mathbb{Q}$ is the set of all rationals and $\times$ is rational multiplication

But $(\mathbb{Q}^*, \times)$, where $\mathbb{Q}^*$ is the set of all non-zero rationals and $\times$ is rational multiplication, is a group!

Examples III

- Prove that $(\mathbb{Z}_p^*, \times)$ is a group when p is a prime, × is integer multiplication $\mod p$, and $\mathbb{Z}_p^* = \{1, \ldots, p-1\}$
- Prove that $(\mathbb{Z}_n^*, \times)$ is $\underline{\text{NOT}}$ a group when n is $\underline{\text{NOT}}$ a prime, × is integer multiplication $\mod n$, and $\mathbb{Z}_n^* = \{1, \ldots, n-1\}$

Examples IV

Groups need not be commutative.

- Define a group that is not commutative. Hint: Consider G as the set of $n \times n$ full-rank matrices with elements in $\mathbb{Q}$. Now, define $\circ$ as matrix multiplication.
  We shall define left and right inverses and left and right identities in the homework. We shall prove interesting properties regarding these inverses and identities.

Generator I

- Consider the group $(\mathbb{Z}_5, +)$
- Note that
    - 2 added 0-times is 0
    - 2 added 1-times is 2
    - 2 added 2-times is 4
    - 2 added 3-times is 1
    - 2 added 4-times is 3
    - 2 added 5-times is 0
    - (and so on)
- We say that 2 generates $(\mathbb{Z}_5, +)$ because we can generate the entire set $\mathbb{Z}_5$ be repeatedly "$+$"-ing 2 to itself
- Consider the group $(\mathbb{Z}_7^*, \times)$. Which elements in $\mathbb{Z}_7$ generate the group? And which elements do not generate the group?

Generator II

- We will introduce a shorthand. By a^k, we represent the number $\overbrace{a \circ a \circ \cdots \circ a}^{k\text{-times}}$
- We define $a^0 = e$, the identity of the group

Repeated Squaring Technique

Let g be a generator of a group $(G, \circ)$. Consider the following algorithm.

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Let $n[0] := g$, the identity of $(G, \circ)$
For $i = 1$ to $k$, do the following:
    $n[i] := n[i-1] \circ n[i-1]$
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- At the termination of the algorithm, we have the following $n[0] = g$, $n[1] = g^2$, $n[2] = g^4$, $\ldots$, $n[k] = g^{2^k}$
- Note that we only used the $\circ$ operation only k times in this algorithm to generate this sequence
- Let i be an integer in the range $\{0, \ldots, 2^{k+1} - 1\}$
- How to compute g^i using $(k+1)$ additional $\circ$ operations?
- Note: This gives us an algorithm to compute g^i, where $i \in \{0, \ldots, 2^{k+1} - 1\}$ using at most $(2k+1)$ $\circ$ operations!

Why Repeated Squaring is Efficient?

- Let $(G, \circ)$ be a group generated by g
- Suppose we are interested in computing g^i
- First Algorithm: Multiply g i-times to get g^i. This method takes $O(i)$ time.
- Second Algorithm: Use repeated squaring to compute g^i. This method takes $O(\log i)$ time.
- Why is the first algorithm an exponential-time algorithm? Why is the second algorithm a polynomial-time algorithm?

Fields

Definition

A field is defined by a set of elements $\mathbb{F}$, and two operators $+$ and $\cdot$. The field $(\mathbb{F}, +, \cdot)$ satisfies the following properties

1. **Closure.** For all $a, b \in \mathbb{F}$, we have $a + b \in \mathbb{F}$ and $a \cdot b \in \mathbb{F}$
2. **Associativity.** For all $a, b, c \in \mathbb{F}$, we have $(a + b) + c = a + (b + c)$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
3. **Commutativity.** For all $a, b \in \mathbb{F}$, we have $a + b = b + a$ and $a \cdot b = b \cdot a$
4. **Additive and Multiplicative identities.** There exists elements $0 \in \mathbb{F}$ and $1 \in \mathbb{F}$ such that for all $a \in \mathbb{F}$, we have $a + 0 = a$ and $a \cdot 1 = a$
5. **Additive inverse.** Every $a \in \mathbb{F}$ has $(-a) \in \mathbb{F}$ such that $a + (-a) = 0$
6. **Multiplicative inverse.** Every $0 \neq a \in \mathbb{G}$ has $(a^{-1}) \in \mathbb{F}$ such that $a \cdot (a^{-1}) = 1$
7. **Distributivity.** For all $a, b, c \in \mathbb{F}$, we have $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Examples

- $(\mathbb{Z}_p, +, \times)$ is a field when p is a prime, $+$ is integer addition mod p, and $\times$ is integer multiplication $\mod p$
- $(\mathbb{Q}, +, \times)$ is a field
- The first example mentioned above is a *finite* field, and the second example mentioned above is an *infinite* field
- Size of any finite field is p^n, where p is a prime and n is a natural number
- Additional Reading: If interested, read about how the fields of size p^2, p^3, $\ldots$ are defined