

Homework 2

1. **Some properties of (\mathbb{Z}_p^*, \times) (25 points).** Recall that \mathbb{Z}_p^* is the set $\{1, \dots, p-1\}$ and \times is integer multiplication mod p , where p is a prime. For example, if $p = 5$, then 2×3 is 1. In this problem, we shall prove that (\mathbb{Z}_p^*, \times) is a group when p is any prime. The only part missing in the lecture was the proof that every $x \in \mathbb{Z}_p^*$ has an inverse. We will find the inverse of any element $x \in \mathbb{Z}_p^*$.

- (a) (10 points) Recall $\binom{p}{k} := \frac{p!}{k!(p-k)!}$. For a prime p , prove that p divides $\binom{p}{k}$, if $k \in \{1, 2, \dots, p-1\}$.

Solution.

- (b) (10 points) Recall that $(1 + x)^p = \sum_{k=0}^p \binom{p}{k} x^k$. Prove by induction on x that, for any $x \in \mathbb{Z}_p^*$, we have

$$\overbrace{x \times x \times \cdots \times x}^{p\text{-times}} = x$$

Solution.

(c) (5 points) For $x \in \mathbb{Z}_p^*$, prove that the inverse of $x \in \mathbb{Z}_p^*$ is given by

$$\overbrace{x \times x \times \cdots \times x}^{(p-2)\text{-times}}$$

That is, prove that $x^{p-1} = 1 \pmod p$, for any prime p and $x \in \mathbb{Z}_p^*$.

Solution.

2. **Understanding Groups: Part one (30 points).** Recall that when we defined a group (G, \circ) , we stated that there exists an element e such that for all $x \in G$ we have $x \circ e = x$. Note that e is “applied on x from the right.” Similarly, for every $x \in G$, we are guaranteed that there exists $\text{inv}(x) \in G$ such that $x \circ \text{inv}(x) = e$. Note that $\text{inv}(x)$ is again “applied to x from the right.”

In this problem, however, we shall explore the following questions: (a) Is there an “identity from the left?” and (b) Is there an “inverse from the left?”

We shall formalize and prove these results in this question.

- (a) (5 points) Prove that it is impossible that there exists $a, b, c \in G$ such that $a \neq b$ but $a \circ c = b \circ c$.

Solution.

(b) (6 points) Prove that $e \circ x = x$, for all $x \in G$.

Solution.

- (c) (6 points) Prove that if there exists an element $\alpha \in G$ such that for **some** $x \in G$, we have $\alpha \circ x = x$, then $\alpha = e$. (Remark: Note that these two steps prove that the “left identity” is identical to the right identity e .)

Solution.

(d) (8 points) Prove that $\text{inv}(x) \circ x = e$.

Solution.

- (e) (5 points) Prove that if there exists an element $\alpha \in G$ and $x \in G$ such that $\alpha \circ x = e$, then $\alpha = \text{inv}(x)$.

(Remark: Note that these two steps prove that the “left inverse of x ” is identical to the right inverse $\text{inv}(x)$.)

Solution.

3. **Understanding Groups: Part Two (15 points).** In this part, we will prove a crucial property of inverses in groups – they are unique. And finally, using this property, we will prove a result that is crucial to the proof of security of one-time pad over the group (G, \circ) .

- (a) (9 points) Suppose $a, b \in G$. Let $\text{inv}(a)$ and $\text{inv}(b)$ be the inverses of a and b , respectively (i.e., $a \circ \text{inv}(a) = e$ and $b \circ \text{inv}(b) = e$). Prove that $\text{inv}(a) = \text{inv}(b)$ if and only if $a = b$.

Solution.

- (b) (6 points) Suppose $m \in G$ is a message and $c \in G$ is a cipher text. Prove that there exists a unique $sk \in G$ such that $m \circ sk = c$.

Solution.

4. **Calculating Large Powers mod p (15 points).** Recall that we learned the repeated squaring algorithm in class. Calculate the following using this concept

$$11^{2024^{2024}+2024} \pmod{101}$$

(Hint: Note that 101 is a prime number and before applying repeated squaring algorithm try to simplify the problem using what you learned in part C of question 1).

Solution.

5. **Practice with Fields (20 points).** We shall work over the field $(\mathbb{Z}_5, +, \times)$.

- (a) (5 points) Addition Table. The (i, j) -th entry in the table is $i + j$. Complete this table. You do not need to fill the black cells because the addition is commutative.

	0	1	2	3	4
0					
1					
2					
3					
4					

Table 1: Addition Table.

- (b) (5 points) Multiplication Table. The (i, j) -th entry in the table is $i \times j$. Complete this table.

	0	1	2	3	4
0					
1					
2					
3					
4					

Table 2: Multiplication Table.

- (c) (5 points) Additive and Multiplicative Inverses. Write the additive and multiplicative inverses in the table below.

	0	1	2	3	4
Additive Inverse					
Multiplicative Inverse					

Table 3: Additive and Multiplicative Inverses Table.

(d) (5 points) Division Table. The (i, j) -th entry in the table is i/j . Complete this table.

	1	2	3	4
0				
1				
2				
3				
4				

Table 4: Division Table.

6. **Order of an Element in (\mathbb{Z}_p^*, \times) . (20 points)** The *order* of an element x in the multiplicative group (\mathbb{Z}_p^*, \times) is the smallest positive integer h such that $x^h = 1 \pmod p$. For example, the order of 2 in (\mathbb{Z}_5^*, \times) is 4, and the order of 4 in (\mathbb{Z}_5^*, \times) is 2.

(a) (5 points) What is the order of 5 in (\mathbb{Z}_7^*, \times) ?

Solution.

(b) (10 points) Let x be an element in (\mathbb{Z}_p^*, \times) such that $x^n = 1 \pmod p$ for some positive integer n and let h be the order of x in (\mathbb{Z}_p^*, \times) . Prove that h divides n .

Solution.

(c) (5 points) Let h be the order of x in (\mathbb{Z}_p^*, \times) . Prove that h divides $(p - 1)$.

Solution.

7. **Defining Multiplication over \mathbb{Z}_{27}^* (25 points).** In the class, we had considered the group $(\mathbb{Z}_{26}, +)$ to construct a one-time pad for one alphabet message. Can we define a group with 26 elements using a “multiplication”-like operation? This problem shall assist you to define the $(\mathbb{Z}_{27}^*, \times)$ group that has 26 elements.

The first attempt from class. Recall that in the class, we had seen that the following is also a group.

$$(\mathbb{Z}_{27} \setminus \{0, 3, 6, 9, 12, 15, 18, 21, 24\}, \times),$$

where \times is integer multiplication mod 27. However, the set had only 18 elements.

In this problem, we shall define $(\mathbb{Z}_{27}^*, \times)$ in a different manner such that the set has 26 elements.

A new approach. Interpret \mathbb{Z}_{27}^* as the set of all triplets (a_0, a_1, a_2) such that $a_0, a_1, a_2 \in \mathbb{Z}_3$ and at least one of them is non-zero. Intuitively, you can think of the triplets as the ternary representation of the elements in \mathbb{Z}_{27}^* . We interpret the triplet (a_0, a_1, a_2) as the polynomial $a_0 + a_1X + a_2X^2$. So, every element in \mathbb{Z}_{27}^* has an associated non-zero polynomial of degree at most 2, and every non-zero polynomial of degree at most 2 has an element in \mathbb{Z}_{27}^* associated with it.

The multiplication (\times operator) of the element (a_0, a_1, a_2) with the element (b_0, b_1, b_2) is defined as the element corresponding to the polynomial

$$(a_0 + a_1X + a_2X^2) \times (b_0 + b_1X + b_2X^2) \pmod{2 + 2X + X^3}$$

The multiplication (\times operator) of the element (a_0, a_1, a_2) with the element (b_0, b_1, b_2) is defined as follows.

Input (a_0, a_1, a_2) and (b_0, b_1, b_2) .

- Define $A(X) := a_0 + a_1X + a_2X^2$ and $B(X) := b_0 + b_1X + b_2X^2$
- Compute $C(X) := A(X) \times B(X)$ (interpret this step as “multiplication of polynomials with integer coefficients”)
- Compute $R(X) := C(X) \pmod{2 + 2X + X^3}$ (interpret this as step as taking a remainder where one treats both polynomials as polynomials with integer coefficients). Let $R(X) = r_0 + r_1X + r_2X^2$
- Return $(c_0, c_1, c_2) = (r_0 \pmod 3, r_1 \pmod 3, r_2 \pmod 3)$

For example, the multiplication $(0, 1, 1) \times (1, 1, 2)$ is computed in the following way.

- $A(X) = X + X^2$ and $B(X) = 1 + X + 2X^2$.
- $C(X) = X + 2X^2 + 3X^3 + 2X^4$.
- $R(X) = -6 - 9X - 2X^2$.
- $(c_0, c_1, c_2) = (0, 0, 1)$.

According to [this definition](#) of the \times operator, solve the following problems.

- (5 points) Evaluate $(1, 1, 1) \times (1, 0, 1)$.

Solution.

- (10 points) Note that $e = (1, 0, 0)$ is an identity element. Find the inverse of $(0, 1, 1)$.
Solution.

- (10 points) Assume that $(\mathbb{Z}_{27}^*, \times)$ is a group. Find the order of the element $(1, 1, 0)$.
(Recall that, in a group (G, \circ) , the order of an element $x \in G$ is the smallest positive integer h such that $\overbrace{x \circ x \circ \cdots \circ x}^{h\text{-times}} = e$)

Solution.

8. **Elliptic curve (10 points).** In class, we have briefly discussed elliptic curve. Here we will see some concrete examples of elliptic curve on finite prime fields.

(a) (5 points). Let $Y^2 = X^3 + X$ be an elliptic curve over the field $(F_{23}, +, \cdot)$. A point (X, Y) lies on the elliptic curve if it satisfies the equation $Y^2 = X^3 + X$.

i. (2 points) Verify that the two points $P = (21, 6)$ and $Q = (18, 10)$ are on the curve.

Solution.

ii. (3 points) Find the point R where the line connecting P and Q intersects the elliptic curve $Y^2 = X^3 + X$. For $R = (x, y)$, define the “inverse of R ” to be the point $S = (x, -y)$. Find the inverse of point R . Recall from the lecture that “ $P + Q$ ” is defined to be the point $S :=$ “inverse of R .”

Solution.

(b) (5 points). Let $Y^2 = X^3 + X + 7$ be an elliptic curve over the field $(F_{17}, +, \cdot)$.

i. (2 points) Verify that the two points $P = (5, 16)$ and $Q = (1, 3)$ are on the curve.

Solution.

ii. (3 points) Find the point R where the line connecting P and Q intersects the elliptic curve $Y^2 = X^3 + X + 7$. Find the inverse of point R .

Solution.

Collaborators :