

## Lecture 03: Groups and Fields

## Definition

A *group*, represented by  $(G, \circ)$ , is defined by a set  $G$  and a binary operator  $\circ$  that satisfies the following properties

- 1 **Closure.** For all  $a, b \in G$ , we have  $a \circ b \in G$
- 2 **Associativity.** For all  $a, b, c \in G$ , we have  $(a \circ b) \circ c = a \circ (b \circ c)$
- 3 **Identity.** There exists an element  $e \in G$  such that for all  $a \in G$ , we have  $a \circ e = a$
- 4 **Inverse.** For every element  $a \in G$ , there exists an element  $(-a) \in G$  such that  $a \circ (-a) = e$

# A Quick Check

- Verify that  $(\{0, 1\}^n, \oplus)$ , where  $\oplus$  is the bit-wise XOR of bits, is a group
  - Closure and Associativity is trivial to verify
  - Show that  $\underbrace{00 \cdots 0}_{n\text{-times}}$  is the identity
  - Show that for  $a \in \{0, 1\}^n$ , the inverse of  $a$  is  $a$  itself

# One-time Pad extended to Arbitrary Groups

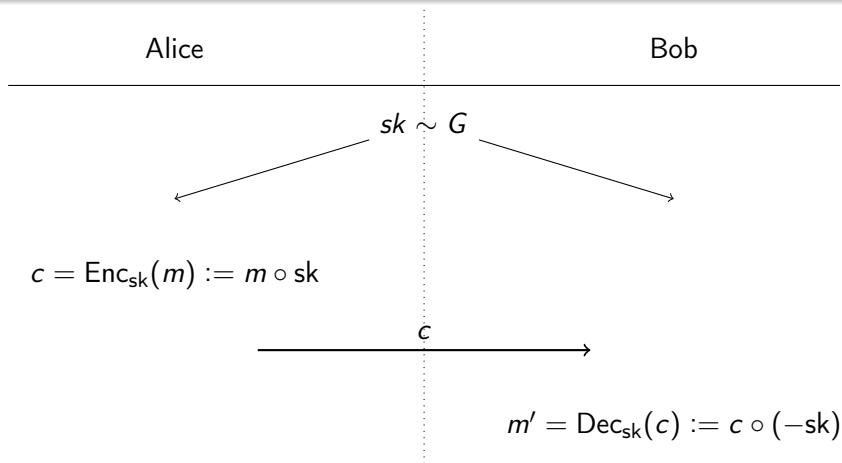


Figure: One-time Pad encryption scheme for the group  $(G, \circ)$ .

Verify that the scheme is always correct

- Groups can have infinite size.  $(\mathbb{Z}, +)$ , where  $\mathbb{Z}$  is the set of all integers and  $+$  is integer addition, is a group (Verify that it satisfies all properties of a group)
- Groups can have finite size.  $(\mathbb{Z}_n, +)$ , where  $\mathbb{Z}_n = \{0, \dots, n-1\}$  and  $+$  is integer addition mod  $n$ , is a group (Verify that it satisfies all properties of a group)

Following are NOT groups. Find which rule is violated.

- $(\mathbb{Z}, \times)$ , where  $\times$  is the integer multiplication
- $(\mathbb{Z}^*, \times)$ , where  $\mathbb{Z}^*$  is the set of all non-zero integers and  $\times$  is the integer multiplication
- $(\mathbb{Q}, \times)$ , where  $\mathbb{Q}$  is the set of all rationals and  $\times$  is rational multiplication

But  $(\mathbb{Q}^*, \times)$ , where  $\mathbb{Q}^*$  is the set of all non-zero rationals and  $\times$  is rational multiplication, is a group!

- Prove that  $(\mathbb{Z}_p^*, \times)$  is a group when  $p$  is a prime,  $\times$  is integer multiplication mod  $p$ , and  $\mathbb{Z}_p^* = \{1, \dots, p-1\}$
- Prove that  $(\mathbb{Z}_n^*, \times)$  is NOT a group when  $n$  is NOT a prime,  $\times$  is integer multiplication mod  $n$ , and  $\mathbb{Z}_n^* = \{1, \dots, n-1\}$

Groups need not be commutative.

- Define a group that is not commutative. Hint: Consider  $G$  as the set of  $n \times n$  full-rank matrices with elements in  $\mathbb{Q}$ . Now, define  $\circ$  as matrix multiplication.

We shall define left and right inverses and left and right identities in the homework. We shall prove interesting properties regarding these inverses and identities.



- Consider the group  $(\mathbb{Z}_5, +)$
- Note that
  - 2 added 0-times is 0
  - 2 added 1-times is 2
  - 2 added 2-times is 4
  - 2 added 3-times is 1
  - 2 added 4-times is 3
  - 2 added 5-times is 0
  - (and so on)
- We say that 2 generates  $(\mathbb{Z}_5, +)$  because we can generate the entire set  $\mathbb{Z}_5$  by repeatedly “+”-ing 2 to itself
- Consider the group  $(\mathbb{Z}_7^*, \times)$ . Which elements in  $\mathbb{Z}_7$  generate the group? And which elements do not generate the group?

- We will introduce a shorthand. By  $a^k$ , we represent the number  $\overbrace{a \circ a \circ \cdots \circ a}^{k\text{-times}}$
- We define  $a^0 = e$ , the identity of the group

# Repeated Squaring Technique

Let  $g$  be a generator of a group  $(G, \circ)$ . Consider the following algorithm.

- Let  $n[0] := g$ , the identity of  $(G, \circ)$
- For  $i = 1$  to  $k$ , do the following:
  - $n[i] := n[i - 1] \circ n[i - 1]$

- At the termination of the algorithm, we have the following  $n[0] = g, n[1] = g^2, n[2] = g^4, \dots, n[k] = g^{2^k}$
- Note that we only used the  $\circ$  operation only  $k$  times in this algorithm to generate this sequence
- Let  $i$  be an integer in the range  $\{0, \dots, 2^{k+1} - 1\}$
- How to compute  $g^i$  using  $(k + 1)$  additional  $\circ$  operations?
- Note: This gives us an algorithm to compute  $g^i$ , where  $i \in \{0, \dots, 2^{k+1} - 1\}$  using at most  $(2k + 1)$   $\circ$  operations!

# Why Repeated Squaring is Efficient?

- Let  $(G, \circ)$  be a group generated by  $g$
- Suppose we are interested in computing  $g^i$
- First Algorithm: Multiply  $g$   $i$ -times to get  $g^i$ . This method takes  $O(i)$  time.
- Second Algorithm: Use repeated squaring to compute  $g^i$ . This method takes  $O(\log i)$  time.
- Why is the first algorithm an exponential-time algorithm?  
Why is the second algorithm a polynomial-time algorithm?

## Definition

A field is defined by a set of elements  $\mathbb{F}$ , and two operators  $+$  and  $\cdot$ . The field  $(\mathbb{F}, +, \cdot)$  satisfies the following properties

- 1 **Closure.** For all  $a, b \in \mathbb{F}$ , we have  $a + b \in \mathbb{F}$  and  $a \cdot b \in \mathbb{F}$
- 2 **Associativity.** For all  $a, b, c \in \mathbb{F}$ , we have  $(a + b) + c = a + (b + c)$  and  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- 3 **Commutativity.** For all  $a, b \in \mathbb{F}$ , we have  $a + b = b + a$  and  $a \cdot b = b \cdot a$
- 4 **Additive and Multiplicative identities.** There exists elements  $0 \in \mathbb{F}$  and  $1 \in \mathbb{F}$  such that for all  $a \in \mathbb{F}$ , we have  $a + 0 = a$  and  $a \cdot 1 = a$
- 5 **Additive inverse.** Every  $a \in \mathbb{F}$  has  $(-a) \in \mathbb{F}$  such that  $a + (-a) = 0$
- 6 **Multiplicative inverse.** Every  $0 \neq a \in \mathbb{G}$  has  $(a^{-1}) \in \mathbb{F}$  such that  $a \cdot (a^{-1}) = 1$
- 7 **Distributivity.** For all  $a, b, c \in \mathbb{F}$ , we have  $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

# Examples

- $(\mathbb{Z}_p, +, \times)$  is a field when  $p$  is a prime,  $+$  is integer addition mod  $p$ , and  $\times$  is integer multiplication mod  $p$
- $(\mathbb{Q}, +, \times)$  is a field
- The first example mentioned above is a *finite* field, and the second example mentioned above is an *infinite* field
- Size of any finite field is  $p^n$ , where  $p$  is a prime and  $n$  is a natural number
- Additional Reading: If interested, read about how the fields of size  $p^2, p^3, \dots$  are defined