An SDP Formulation for Minimizing p-th Order Controversy with Unknown Initial Opinions

Meher Chaitanya[1], Adarsh Barik[2], and Jean Honorio[3]

[1] ETH Zürich, Social Networks Lab, Switzerland
mpindiprolu@ethz.ch
[2] National University of Singapore, Singapore
[3] University of Melbourne, Australia

Abstract. In this paper, we investigate the problem of minimizing p-th order controversy within a network, assuming a framework of opinion evolution based on the well-established Friedkin-Johnsen (FJ) model. We define p-th order controversy as $f_p(L) = s^T(I+L)^{-p}s$, where s represents the vector of users' fixed albeit *undisclosed* initial opinions, I is the identity matrix, and L is the graph Laplacian associated with the underlying network. Notably, for the case of $p = 1$, this function transforms into the widely recognized polarization-disagreement index, and for $p = 2$, it aligns with the standard polarization [1,2]. We focus on minimizing $f_p(L)$ within a novel and realistic framework, where users' initial opinions s are undisclosed. Due to the undisclosed nature of users' initial opinions, achieving the exact minimization of $f_p(L)$ proves unattainable within our innovative and practical framework. To address this challenge, we introduce a novel semidefinite programming formulation designed to enable the minimization of the upper bound of $f_p(L)$ without the need for knowledge of initial opinions. Furthermore, our empirical findings demonstrate its effectiveness, surpassing current state-of-the-art methodologies.

Keywords: Controversy, Polarization, Polarization-Disagreement, Friedkin-Johnsen Dynamics

1 Introduction

Polarization within social networks denotes the emergence and intensification of divergent opinions or perspectives within a community, signifying the creation of distinct clusters comprised of like-minded individuals. Paradoxically, the intended goal of social media to foster closeness among individuals has resulted in the divergence of opinions [3]. This phenomenon results in diminished interaction and understanding between various ideological groups as the individuals are exposed to others who reinforce their existing beliefs [4]. Disagreement, an integral element of polarization, represents the discord and divergence in opinions among members of the network. In social networks, the amplification of polarization often stems from echo chambers and filter bubbles, where individuals are exposed predominantly to information that aligns with their existing beliefs [5]. Understanding the dynamics of polarization and disagreement is essential for devising strategies that promote diverse perspectives and mitigate the negative impacts of filter bubbles in online social spaces.

Notation: Let $G = (V, E, W)$ be an undirected network with the vertex set $V = \{1, \ldots, n\}$ and the edge set, $E \subseteq \binom{n}{2}$. The natural and real numbers set is denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. The cardinality of the set V and E is denoted by n and m, respectively. Let s_i denote the i^{th} entry of vector s. Every real symmetric matrix $M \in R^{n\times n}$ can be decomposed as $M = U\Lambda U^T$ where $U \in R^{n\times n}$ is orthonormal (i.e., $U^TU = UU^T = I$) and Λ is a diagonal matrix, whose diagonal elements are the corresponding eigenvalues $\Lambda_{ii} = \lambda_i$, $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. A matrix M is normal if it commutes with its conjugate transpose, i.e., $M^TM = MM^T$. A symmetric matrix is positive definite (PD) if all its eigenvalues are positive (i.e., $\lambda > 0$). The set PD and PSD matrices are denoted by S^n_{++} and S^n_{+}, respectively. $\mathrm{vec}(A)$ represents the vectorized form of the matrix A. The algebraic connectivity of a given Laplacian matrix is provided by its second smallest eigenvalue, λ_2. We use Tr to denote the trace of the matrix. For a vector s, $\|s\|_1$ and $\|s\|_2$ denote the ℓ_1 and ℓ_2 norm respectively. Unless required, we omit the dimensions when they are clear from the context.

1.1 Polarization and Controversy

In this paper, we assume that the underlying opinions adhere to one of the prominent averaging models, Friedkin-Johnsen's (FJ) opinion formulation model [6], which integrates individuals' initial opinions into the evolutionary process. The FJ model is a popular averaging model that incorporates individuals' prejudices or initial opinions within the network. Let $s \in \mathbb{R}^n$ represent the immutable innate opinions of actors in the network. The expressed opinions are denoted by $z \in \mathbb{R}^n$. Let $w_{ij} \geq 0$ represent the weight on the edge $(i, j) \in E$. The fixed point iteration of the Friedkin-Johnsen opinion dynamics model is then expressed as

$$z_i^{(t)} = \frac{s_i + \sum_{j\in N(i)} w_{ij} z_j^{(t-1)}}{\sum_{j\in N(i)} w_{ij} + 1} .$$

At each time step, each actor embraces an expressed opinion proportionate to the average of its individual innate opinion and the opinions of its neighbors. It is established that the Friedkin-Johnsen dynamics, as defined above, converge to a set of equilibrium opinions z^* [7], as given by: $z^* = (I + L)^{-1}s$, where $L = D - A$ is the graph Laplacian - with D, a diagonal matrix of (weighted) degrees associated with each node and A is the (weighted) adjacency matrix) - and I is the identity matrix. Note that $(I + L)$ is a positive definite matrix (i.e., all the eigenvalues are positive), ensuring the existence of its inverse. It can be observed that z_i represents a convex combination of innate opinions of all nodes, including node i in the network. Now, we formally define the p-th order polarization.

Definition 1 (p-th Order Controversy). *For a network, G with innate opinions denoted by s, and the expressed opinions z, the p-th order controversy is defined as*

$$f_p(L) = s^T(I + L)^{-p}s \tag{1}$$

Definition 2 (Polarization). *The function $f_p(L)$, for $p = 2$, is called polarization ($\mathcal{P}(z)$) and is defined as the measure of variation in expressed opinions among the* *users in the network. It is calculated as follows:*

$$\mathcal{P}(z) = z^T z = s^T (I + L)^{-2} s = f_2(L) \tag{2}$$

Chen et al. [2] defined the above equation as *controversy*. $\mathcal{P}(z)$ is known to be non-convex [8]. The disagreement in the network reflects the difference in the expressed opinion of a user/node with neighbors. For a vector of expressed opinions, z, the disagreement for the given network G is defined as $\mathcal{D}(z) = \sum_{(i,j)\in E} w_{ij}(z_i - z_j)^2 = z^T L z = s^T (I + L)^{-1} L (I + L)^{-1} s$.

Definition 3 (Polarization-Disagreement). *In equation* (1)*, when* $p = 1$*, the function* $f_1(L)$ *evaluates to polarization-disagreement index [1,2] and is given by:*

$$\mathcal{P}(z) + \mathcal{D}(z) = s^T (I + L)^{-1} s = f_1(L). \tag{3}$$

Polarization-Disagreement index is a convex function [9].

p-th order controversy (for $p \geq 3$) arises in computing multi-period scenarios, where controversy is assessed across different time epochs. In multi-period controversy, the final opinions (z) of one epoch become initial opinions (s) of the next, i.e., $s_t = z_{t-1}$, where t is the epoch or time step. Thus, for epoch t, the final opinion becomes $z_t = (I + L)^{-t} s$, and the corresponding polarization is $z_t^T z_t = s^T (I + L)^{-2t} s$, which can be regarded as p-th order polarization, where $p = 2t$.

Definition 4 (Average Conflict Risk (ACR)). *Assuming that the initial opinions are sampled from a uniform distribution, the Average Conflict Risk (ACR) for* p*-th order polarization is defined by taking the expectation of all possible initial opinions and is defined as follows:*

$$ACR = E[s^T (I + L)^{-p} s] = E[Tr(s^T (I + L)^{-p} s)] = E[Tr(ss^T (I + L)^{-p})] \tag{4}$$

$$= Tr[E(ss^T)(I + L)^{-p}] = Tr((I + L)^{-p}) \ . \tag{5}$$

Thus the ACR for polarization ($p = 2$) and polarization-disagreement ($p = 1$) is given by $\mathrm{Tr}((I + L)^{-2})$ and $\mathrm{Tr}((I + L)^{-1})$ respectively [2].

In this research, we aim to investigate how a social networking platform administrator can alter the network topology within a specified budget to minimize $f_p(L)$. While expressed or external opinions can be empirically quantified, a significant constraint of this model is the near impossibility of prior knowledge about initial opinions. In our study, we posit that the network administrator is only aware of opinionated clusters but lacks access to individuals' initial opinions. Towards this, in this paper, we are interested in minimizing the following:

Definition 5 (Minimizing p-th order Polarization). *Given an adjacency matrix* A_0 *and a budget* k*, the network administrator, being agnostic to the initial opinions* s*, aims to find an adjacency matrix* A *and its associated graph Laplacian* $L = D - A$ *with expressed opinions* z *that has minimum* $f_p(L)$*. Mathematically, it is expressed as:*

$$\begin{aligned} \min_{A} \quad & f_p(L) = s^T (I + L)^{-p} s \\ \textit{subject to} \quad & \| \mathrm{vec}(A) - \mathrm{vec}(A_0) \|_1 \leq 2k \ . \end{aligned} \tag{6}$$

Current methodologies for minimizing $f_p(L)$ (equation 6) primarily revolve around the premise of having insight into the distribution of initial opinion vectors, as described in Definition 4 of ACR and in Section 2. However, in this paper, we operate under the assumption of lacking any knowledge about the distribution of initial opinions s.

2 Related Work

In recent years, there has been a surge in formulating and exploring optimization problems related to opinion dynamics models, each addressing diverse objectives. A substantial body of research has concentrated on enhancing collective opinion by employing various strategies. These strategies encompass adjustments to actors' initial opinions [10] and the modulation of individuals' susceptibility to persuasion [11,12]. An increasingly pivotal area of investigation involves the optimization of social phenomena within networks. [1] tackled the challenge of identifying an undirected graph topology with a specified edge cardinality, aiming to minimize both polarization and disagreement. Their study showcased that minimizing the Polarization-Disagreement index in a network is convex concerning the Laplacian matrix L. Furthermore, they established the existence of a graph topology with $\mathcal{O}(\frac{n}{\epsilon^2})$ edges, ensuring a $(1+\epsilon)$ approximation to the optimal solution. Additionally, the authors presented a polynomial-time semidefinite programming approach designed to optimize the same objective through the perturbation of initial opinions.

[13] provided a scalable greedy algorithm for optimizing the Polarization - Disagreement index within a specified budget based on a given set of initial opinions. Despite the non-submodular nature of the function, they successfully established a bounded approximation ratio. To address the issue of potential echo chambers, [14] augmented the FJ model by connecting edges between users with matching ideologies. They observed that this augmentation could lead to potential echo chambers. Examining the adversarial scenario, [15] investigated how Disagreement and Polarization can be increased by perturbing the initial opinions of nodes in the network. A recent study by [8] delved into how an administrator or a centralized planner can alter the network to reduce polarization. They addressed this problem in scenarios where initial opinions were known and unknown. When the planner is oblivious to initial opinions, they propose the Fiedler difference vector approach (FD) for reducing polarization.

Need for alternative approach: Most existing studies aiming to minimize polarization and polarization-disagreement indices by altering network topology assume complete knowledge of initial opinions. The only methodologies in the literature that remain indifferent to the selection of initial opinions are the Average Conflict Risk (ACR) and the Fiedler difference vector (FD). ACR assumes that the covariance matrix of initial opinions, $\langle s, s^T \rangle$, is an identity matrix, and FD operates by heuristically minimizing the λ_2 of the graph Laplacian. In real-world networks, the initial opinions need not follow a particular distribution, and in Section 4, we empirically demonstrate that all eigenvalues play a pivotal role in minimizing polarization. To address the limitations of existing methods, we propose a semidefinite programming approach that assumes no knowledge of initial opinions. We use ACR and FD as benchmark references for comparison in Section 4.

3 Semidefinite Programming formulation to Minimize Polarization

This section proposes a novel semidefinite programming formulation for minimizing the p-th order polarization defined in Definition 1. Our approach begins with a broad formulation, addressing the optimization problem outlined below (equation 7 transforms into minimizing polarization when $p = 2$ and polarization-disagreement when $p = 1$):

$$\begin{aligned} \min_{L} \quad & f_p(L) = s^T (I + L)^{-p} s \\ \text{subject to} \quad & L \in \mathcal{L} \\ & \| \operatorname{vec}(L) - \operatorname{vec}(L_0) \|_1 \leq 4k \ , \end{aligned} \tag{7}$$

Here, L_0 represents the Laplacian of the given graph. The budget constraint is adjusted to $4k$ instead of $2k$ as modifying each entry in the adjacency matrix is equivalent to altering 4 entries in the Laplacian. Without knowledge of the initial opinion vector s, resolving this optimization problem in its present state is not feasible. Nevertheless, by leveraging the spectral property of the Laplacian L, and assuming that the initial opinions are mean-centered, i.e., $s \perp \vec{1}$, we can establish the lower bound for the objective function as

$$f_p(L) \geq (\lambda_{min}((I + L)^{-p})) s^T s = \frac{s^T s}{(1 + \lambda_{max}(L))^p} \tag{8}$$

For the upper bound, note that the eigenvector corresponding to the largest eigenvalue of $(I + L)^{-p}$ is $\vec{1}$. Since s is orthogonal to $\vec{1}$, we get

$$f_p(L) \leq (\lambda_{max}((I + L)^{-p})) s^T s = \frac{s^T s}{(1 + \lambda_2(L))^p} \tag{9}$$

For the unconstrained budget, the global optimum for $f_p(L)$ is attained for complete graphs. The rationale behind this is that a complete graph possesses the maximum possible eigenvalue (λ_{max}) of n, and any other Laplacian matrix with n vertices will have an eigenvalue no greater than n. However, this solution becomes impractical when the budget k is limited in a constrained setting.

One approach to minimizing the p-th order polarization is maximizing $\lambda_2(L)$. This strategy of maximizing the spectral gap has been explored to enhance network robustness by introducing additional links, as discussed in [16] and [17]. This heuristic involves adding edges between nonadjacent vertices in the graph with the largest absolute difference $|\mu_i - \mu_j|$ in the entries of the Fiedler vector μ [18]. Miklos et al. [8] used these heuristics to establish bounds on the change in polarization ($p = 2$) resulting from adding an edge in G. However, the spectral gap approach tends to overlook the potential effects of other eigenvalues by concentrating solely on maximizing λ_2 (refer to Section(4) for details).

Here, we introduce a systematic method designed to increase all the eigenvalues of the matrix $L \in \mathcal{L}$. We start with the eigenvalue decomposition of $(I + L) = V \Lambda V^T$ where V is an orthonormal matrix. Observe that columns of V (denoted as v_i) can be treated as the orthonormal basis of $\mathbb{R}^n$ as L is a normal matrix. Thus, s can be represented as a linear combination of columns of V, i.e., $s = \sum_{i=1}^{n} \alpha_i v_i$ and $\|s\| = \sum_{i=1}^{n} \alpha_i^2$. The elements of Λ are expressed as $1 + \lambda_i$ with $\lambda_i \geq 0$ given that $L \in S_+^n$.

$$s^T (I+L)^{-p} s = s^T V \Lambda^{-p} V^T s = \sum_{i=1}^{n} \frac{1}{(1+\lambda_i)^p} (s^T v_i)^2 = \sum_{i=1}^{n} \frac{1}{(1+\lambda_i)^p} \alpha_i^2$$
$$\leq \sum_{i=1}^{n} \frac{1}{(1+\lambda_i)^p} \max_i \alpha_i^2 \leq \sum_{i=1}^{n} \frac{1}{(1+\lambda_i)^p} \|s\|^2,$$

Where the last inequality follows trivially. Considering the network administrator's lack of knowledge of s, it is reasonable to minimize the upper bound. This involves minimizing $M := \sum_{i=1}^{n} \frac{1}{(1+\lambda_i)^p} = \text{Trace}((I+L)^{-p})$ (ACR given in equation 4), a strategy employed in [2] and [1] for $p \in \{1, 2\}$. However, the computational challenges associated with inverting $(I + L)$ and repeated matrix multiplications make the minimization of Trace$((I+L)^{-p})$ challenging.

Essentially, any monotonically increasing function of M can be minimized to achieve an equivalent minimization of M. An illustrative example is $\log M$. Additionally, notice that $M \geq 1$, ensuring $\log M \geq 0$. This observation emphasizes that $M \log M$ is also monotonically increasing, and minimizing $M \log M$ is equivalent to minimizing M. By applying the log-sum inequality, we obtain:

$$\begin{aligned} M \log M &\leq \sum_{i=1}^{n} \frac{1}{(1+\lambda_i)^p} \log \frac{n}{(1+\lambda_i)^p} \leq \sum_{i=1}^{n} \log \frac{n}{(1+\lambda_i)^p} \\ &\leq n \log n - p \log \Pi_{i=1}^{n} (1+\lambda_i) \\ &\leq n \log n - p \log \det(I+L) \end{aligned}$$

Therefore, the minimization of $-p \log \det(I + L)$ effectively minimizes the upper bound on p-th order polarization. Notably, this formulation is centered on augmenting the eigenvalues of L. The convex formulation employing $\log(\det(I + L))$ is provided below. We retain the exponent p in the objective function to emphasize the generalizability of our approach.

$$\begin{aligned} \min_{L} \quad & -p \log(\det(I+L)) \\ \text{subject to} \quad & L \in \mathcal{L} \\ & \| \text{vec}(L) - \text{vec}(L_0) \|_1 \leq 4k \ . \end{aligned} \tag{10}$$

Substantial research has been dedicated to this objective function with a focus on scalability [19], [20], [21], [22].

Ensuring Sparsity: When addressing the convex formulation in (10), the resulting Laplacian may exhibit high density. Even for modest budgets, our observations indicate that the solution to (10) tends to converge to a complete graph with reduced weights distributed across the network. This phenomenon has been empirically noted with the combinatorial graph Laplacian, leading to the introduction of various sparsity-preserving techniques in the literature [23], [24], [25], [26].

Empirical findings suggest that employing an ℓ_1 regularizer on the combinatorial graph Laplacian, coupled with generalized Laplacian constraints pertaining to symmetry and negativity of off-diagonal elements, proves effective in practice [26]. Adhering to this principle, we adopt an entry-wise ℓ_1 norm for the graph Laplacian.

To mitigate the echo chamber effects and reduce polarization, it is beneficial to introduce edges connecting users with opposing opinions. We perform this by establishing two opinionated clusters by grouping users with similar opinions and labeling one cluster as "-1" and the other as "1". In the subsequent formulation, cluster(i) denotes the opinionated cluster of node i (assuming a binary opinionated model). The objective is to bias the optimization problem towards adding edges between opinionated clusters while simultaneously maximizing eigenvalues of L. Otherwise, we penalize it with a penalty, i.e., for two nodes i and j penalty c_{ij} is defined as $c_{ij} = \mid \text{cluster}(i) + \text{cluster}(j) \mid$. The extent of the penalty is determined by $\varphi \in R$. The mathematical formulation with penalization is presented below.

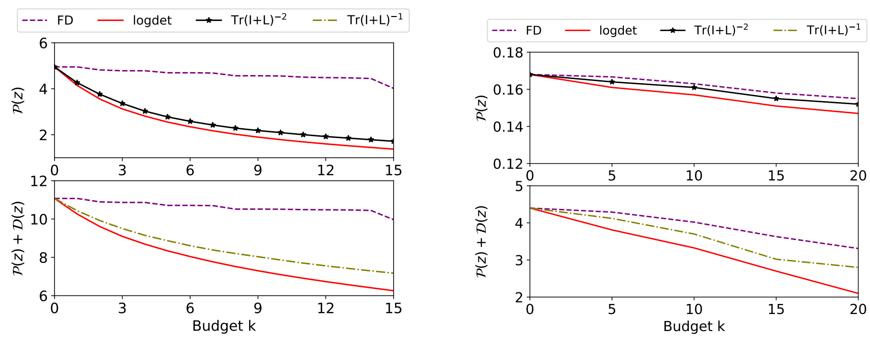
$$\begin{aligned} \min_{L} \quad & -p \log(\det(I+L)) + \varphi \sum_{i<j} c_{ij} |L_{ij}| \\ \text{subject to} \quad & L \in \mathcal{L} \\ & \| \operatorname{vec}(L) - \operatorname{vec}(L_0) \|_1 \leq 4k \ . \end{aligned} \tag{11}$$

The outcome derived from equation (11) is subsequently refined through pruning, employing a threshold ρ to eliminate smaller weights in L by setting them to zero. It is important to note that after pruning, the resulting matrix $\hat{L}$ may not conform to Laplacian properties. To address this, we apply the following projection to obtain the optimal Laplacian L^{proj} that is closest to $\hat{L}$, as outlined in [27]: $L^{proj}_{ii} = -\sum_{j=1, j\neq i}^{n} \hat{L}_{ij}, \forall i \in \{1, \cdots, n\}$. Only the diagonal entries need to be updated after pruning.

4 Empirical Results

In this section, we will conduct an empirical comparison of the performance of our approach (11) with alternative eigenvalue maximization methods, including Trace ((4) for $p \in \{1, 2\}$) and the Fiedler Difference vector (FD) ([17], [8]). Our empirical analysis will focus on minimizing p-th order polarization for $p \in \{1, 2\}$. All three heuristics aim to modify the network's topology with the goal of minimizing both polarization ($\mathcal{P}(z)$ for $p = 2$ in equation (7)) and polarization-disagreement ($\mathcal{P}(z)$ + $\mathcal{D}(z)$ for $p = 1$ in equation (7)) indices (The experiments are run using the CVX [28]). A recent study by Miklos et al. [8] delved into the FD approach within the context of polarization. Their findings indicated that the FD approach does not consistently necessitate the addition of edges between users with opposing opinions to mitigate polarization. They observed that FD reduces polarization without reducing the homophily of the network. We evaluate the efficacy of these heuristics on real-world and synthetic networks. We consider models based on stochastic block and preferential attachment for synthetic networks.

***Karate Club Network**:* This social network captures a clash of opinions involving an instructor and an administrator in a karate club, as outlined by Zachary [29]. Comprising 34 nodes and 78 edges, this undirected network represents club members, with each edge symbolizing a connection between two members. There are two opinionated clusters among club members stemming from the conflict. Each of these clusters is assigned an internal opinion of ”1” or ”-1.” In Figure (1(a), we depict the variation in $\mathcal{P}(z)$ and $\mathcal{P}(z) + \mathcal{D}(z)$ across different budgets. Notably, the $\log\det$ approach substantially reduces polarization and polarization-disagreement indices compared to the Trace (equation 4 for $p \in \{1, 2\}$) and FD methods. As FD achieves polarization reduction by introducing a single edge, it creates the sparsest graph. For the $\log\det$ approach (11) with an absolute value thresholding parameter $\rho = 0.05$ and $\varphi = 0.05$, the average number of non-zero entries in the matrix for budgets $k = 1$ to $k = 15$ is 112.



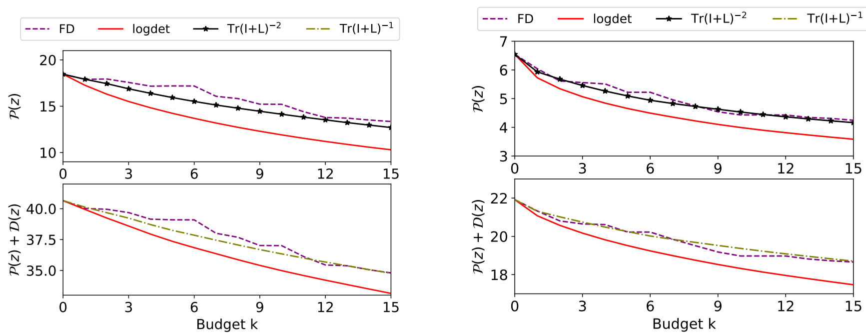
(a) Change in polarization $\mathcal{P}(z)$ and polarization-disagreement $(\mathcal{P}(z) + \mathcal{D}(z))$ with budget on Karate Club network.

(b) Change in polarization $\mathcal{P}(z)$ and polarization-disagreement $(\mathcal{P}(z) + \mathcal{D}(z))$ with budget on Twitter network

Fig. 1: Reduction in Polarization on Karate Club and Twitter Networks

Twitter Network*:* [30] compiled the Twitter dataset for the analysis of the Delhi legislative assembly elections debate. The dataset, obtained through hashtags such as #BJP, #AAP, #Congress, and #Polls2013, forms an undirected network comprising 548 users engaged in 3638 interactions via tweets. Initial opinions are derived from users' interactions on Twitter using sentiment analysis. Given that the opinions acquired are non-binary and continuous, we categorize them into two classes for the computation of equation (11). Continuous opinions are utilized to determine the final polarization. Figure 1(b) visually represents the variations in polarization $(\mathcal{P}(z))$ and polarization-disagreement $(\mathcal{P}(z)+\mathcal{D}(z))$ indices using the $\log\det$ approach (11), the Trace approach (equation 4 for $p \in \{1, 2\}$), and the Fiedler Difference (FD) method. The $\log\det$ approach is more effective in reducing both indices than the alternative methods. In (11), parameters $|\rho|$ and φ are set to 0.02 each. The average number of non-zeros in the resulting matrix across all budgets using the $\log\det$ approach is 4798 (with pruned edges totaling $295, 506$).

Stochastic Block Models*:* The Stochastic Block Model (SBM) is a generative model that produces random graphs containing community structure based on the notion of groups of nodes. We generate two communities with 100 nodes in each of them. The inter-cluster and intra-cluster densities are given by 0.05 and 0.1, respectively. The total number of edges is 1490. We distribute the initial opinions in two different ways. In the first case, we provide an opinion of "-1" to one block of nodes and an opinion of "$+1$" to another. Here, the opinionated clusters are well connected, so the FD approach connects the nodes across two clusters to increase the algebraic connectivity. The penalization term in equation (11) enforces it to connect to nodes across opposite opinionated clusters. The decrease in polarization and polarization-disagreement is shown in Figure 2(a). In the second case, we distribute the opinions of "$+1$" and "-1" uniformly within each block, and the variation in polarization and polarization-disagreement for all three heuristics is shown in Figure 2(b). We observed that $\log\det$ significantly outperforms both Trace and FD approaches in both cases. For both cases we use parameters $|\rho| = 0.02$ and $\varphi = 0.05$.



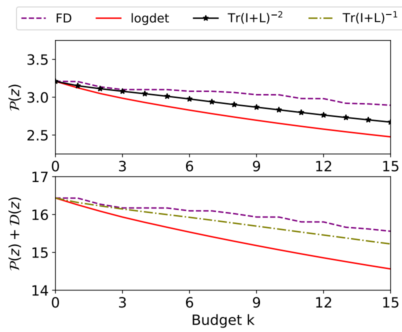
(a) Change in polarization and polarization-disagreement with budget k in SBM when an initial opinion of "-1" is assigned to one community of nodes and an opinion "+1" to the other community.

(b) Change in polarization and polarization-disagreement for uniformly distributed opinions within each community in SBM.

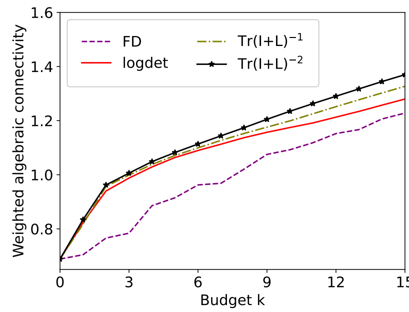
Fig. 2: Reduction in Polarization and Polarization-Disagreement on SBM

Preferential Attachment Models: Preferential attachment (PA) delineates a mechanism of graph evolution wherein nodes with higher degrees exhibit an augmented probability of acquiring new neighbors, a concept tailored to simulate power-law behavior [31]. In our analysis, an incoming vertex establishes connections with a maximum of four other pre-existing vertices within the graph. The resulting PA network comprises 200 nodes and 736 edges. Initial opinions of "1" and "-1" are uniformly assigned at random to nodes in the network. The reduction in $\mathcal{P}(z)$ and $(P)(z) + \mathcal{D}(z)$ spanning budgets $k = 1$ to $k = 15$ is depicted in Figure 3(a). A substantial decrease in polarization is observed when employing the $\log\det$ method in comparison to the FD and Trace approaches. The average number of edges in the network, following $\log\det$ relaxation in (11) with parameters $\varphi = 0.05$ and $|\rho| = 0.04$, amounts to 1154.

Analysis: From the above empirical analysis, we conclude that $\log\det$ (11) outperforms the Fielder Difference and Tr approach in minimizing polarization and polarization-



(a) Reduction in polarization and polarization-disagreement indices with varying budgets using $\log\det$, Trace, and Fielder Difference (FD) approaches on PA network.



(b) Change in algebraic connectivity for uniformly distributed opinions within each block using log(det), Trace and Fielder Difference (FD) approaches on SBM.

Fig. 3: (a) Reduction in Polarization and Polarization-Disagreement on PA network. (b) Variation in Algebraic Connectivity

disagreement indices. If the opposite opinionated clusters are well connected, then there is a considerable reduction in polarization using $\log\det$ compared to FD. The absolute value of thresholding parameter ρ is in the range of $0.01 - 0.05$ (based on empirical observation). Figure 3(b) shows the effect on algebraic connectivity by all three different approaches. It can be observed that minimization of $\mathrm{Tr}((I+L)^{-2})$ considerably reduces the algebraic connectivity of the resultant graph Laplacian compared to both FD and $\log\det$, but it still fails to outperform $\log\det$ in terms of decreasing polarization and polarization-disagreement. This suggests that increasing λ_2 could possibly increase the other eigenvalues due to PSD, but this increase is not significant enough for both FD and Tr to reduce the polarization. Also, with the above empirical analysis, Trace (4) minimizes polarization by increasing the extreme eigenvalues, whereas $\log\det$ minimizes polarization by increasing a subset of the eigenvalues (need not be extreme ones) of L. This shows that all eigenvalues of the network play a pivotal role in reducing polarization.

Conclusion and Future Directions In this paper, we studied the problem of minimizing p-th order polarization by altering the network's topology under the scenario when the network administrator is oblivious to initial opinions. We provide a general computationally tractable semidefinite programming relaxation framework for minimizing the upper bound of this objective function and empirically demonstrate that it outperforms the existing state-of-the-art methodologies.

Current scalability investigations predominantly center around computing the value polarization ($s^T(I+L)^{-2}s$) and polarization-disagreement ($s^T(I+L)^{-1}s$) [32]. In practice, it has been shown that SDPs can be scaled to millions of nodes for problems such as Maximum Cut by employing tight approximations. In the future, it would be interesting to explore the applicability of such concepts in deriving approximation bounds for minimizing p-th order polarization and scaling them to extremely large network configurations.

5 Acknowledgments

This research, in part, is supported by the Swiss National Science Foundation (SNSF Grant Number 209488).

References

1. C. Musco, C. Musco, and C. E. Tsourakakis, "Minimizing polarization and disagreement in social networks," in *Proceedings of the 2018 world wide web conference*, pp. 369–378, 2018.
2. X. Chen, J. Lijffijt, and T. De Bie, "Quantifying and minimizing risk of conflict in social networks," in *Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, pp. 1197–1205, 2018.
3. L. A. Adamic and N. Glance, "The political blogosphere and the 2004 us election: divided they blog," in *Proceedings of the 3rd international workshop on Link discovery*, pp. 36–43, 2005.
4. S. Cass, "The law of group polarization," *Journal of Political Philosophy*, vol. 10, no. 2, pp. 175–195, 2002.
5. D. Kahneman, *Thinking, fast and slow*. macmillan, 2011.
6. N. E. Friedkin and E. C. Johnsen, "Social influence and opinions," *Journal of mathematical sociology*, vol. 15, no. 3-4, pp. 193–206, 1990.
7. D. Bindel, J. Kleinberg, and S. Oren, "How bad is forming your own opinion?," *Games and Economic Behavior*, vol. 92, pp. 248–265, 2015.
8. M. Z. Racz and D. E. Rigobon, "Towards consensus: Reducing polarization by perturbing social networks," *arXiv preprint arXiv:2206.08996*, 2022.
9. D. S. Bernstein, "Matrix mathematics," in *Matrix Mathematics*, Princeton university press, 2009.
10. Y. Yi, T. Castiglia, and S. Patterson, "Shifting opinions in a social network through leader selection," *IEEE Transactions on Control of Network Systems*, vol. 8, no. 3, pp. 1116–1127, 2021.
11. P. Xu, W. Hu, J. Wu, and W. Liu, "Opinion maximization in social trust networks," in *Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI-20*, pp. 1251–1257, International Joint Conferences on Artificial Intelligence Organization, 7 2020.
12. R. Abebe, J. Kleinberg, D. Parkes, and C. E. Tsourakakis, "Opinion dynamics with varying susceptibility to persuasion," in *Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, pp. 1089–1098, 2018.
13. L. Zhu, Q. Bao, and Z. Zhang, "Minimizing polarization and disagreement in social networks via link recommendation," *Advances in Neural Information Processing Systems*, vol. 34, pp. 2072–2084, 2021.
14. U. Chitra and C. Musco, "Analyzing the impact of filter bubbles on social network polarization," in *Proceedings of the 13th International Conference on Web Search and Data Mining*, pp. 115–123, 2020.
15. M. F. Chen and M. Z. Rácz, "An adversarial model of network disruption: Maximizing disagreement and polarization in social networks," *IEEE Transactions on Network Science and Engineering*, vol. 9, no. 2, pp. 728–739, 2021.

16. A. Ghosh and S. Boyd, "Growing well-connected graphs," in *Proceedings of the 45th IEEE Conference on Decision and Control*, pp. 6605–6611, IEEE, 2006.
17. H. Wang and P. Van Mieghem, "Algebraic connectivity optimization via link addition," in *3d International ICST Conference on Bio-Inspired Models of Network, Information, and Computing Systems*, 2010.
18. F. R. Chung, *Spectral graph theory*, vol. 92. American Mathematical Soc., 1997.
19. L. Vandenberghe, S. Boyd, and S.-P. Wu, "Determinant maximization with linear matrix inequality constraints," *SIAM journal on matrix analysis and applications*, vol. 19, no. 2, pp. 499–533, 1998.
20. C. Boutsidis, P. Drineas, P. Kambadur, E.-M. Kontopoulou, and A. Zouzias, "A randomized algorithm for approximating the log determinant of a symmetric positive definite matrix," *Linear Algebra and its Applications*, vol. 533, pp. 95–117, 2017.
21. C. Wang, D. Sun, and K.-C. Toh, "Solving log-determinant optimization problems by a newton-cg primal proximal point algorithm," *SIAM Journal on Optimization*, vol. 20, no. 6, pp. 2994–3013, 2010.
22. M. J. Todd, *Minimum-volume ellipsoids: Theory and algorithms*. SIAM, 2016.
23. H. E. Egilmez, E. Pavez, and A. Ortega, "Graph learning from data under laplacian and structural constraints," *IEEE Journal of Selected Topics in Signal Processing*, vol. 11, no. 6, pp. 825–841, 2017.
24. E. Pavez, H. E. Egilmez, and A. Ortega, "Learning graphs with monotone topology properties and multiple connected components," *IEEE Transactions on Signal Processing*, vol. 66, no. 9, pp. 2399–2413, 2018.
25. E. Pavez and A. Ortega, "Generalized laplacian precision matrix estimation for graph signal processing," in *2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 6350–6354, IEEE, 2016.
26. J. Ying, J. V. de Miranda Cardoso, and D. Palomar, "Nonconvex sparse graph learning under laplacian constrained graphical model," *Advances in Neural Information Processing Systems*, vol. 33, pp. 7101–7113, 2020.
27. K. Sato, "Optimal graph laplacian," *Automatica*, vol. 103, pp. 374–378, 2019.
28. S. Diamond and S. Boyd, "Cvxpy: A python-embedded modeling language for convex optimization," *The Journal of Machine Learning Research*, vol. 17, no. 1, pp. 2909–2913, 2016.
29. W. W. Zachary, "An information flow model for conflict and fission in small groups," *Journal of anthropological research*, vol. 33, no. 4, pp. 452–473, 1977.
30. A. De, S. Bhattacharya, P. Bhattacharya, N. Ganguly, and S. Chakrabarti, "Learning a linear influence model from transient opinion dynamics," in *Proceedings of the 23rd ACM International Conference on Conference on Information and Knowledge Management*, pp. 401–410, 2014.
31. M. Faloutsos, P. Faloutsos, and C. Faloutsos, "On power-law relationships of the internet topology," *ACM SIGCOMM computer communication review*, vol. 29, no. 4, pp. 251–262, 1999.
32. W. Xu, Q. Bao, and Z. Zhang, "Fast evaluation for relevant quantities of opinion dynamics," in *Proceedings of the Web Conference 2021*, pp. 2037–2045, 2021.