Topic 2: Combinatorics Review, Linear Recurrence

Code presentation

Basics

$$C_n^m = C(n,m) = nCm = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

$$P_n^m = P(n,m) = nPm = \frac{n!}{m!}$$

(1) Compute in $O(1)$ with factorials

(2) Pascal's Triangle :

1

1 1

1 2 1

1 3 3 1

. . .

(3) . . .

Pattern Finding

The ability to come up with patterns is important !!!

Common Patterns:

Catalan

Stirling I / II

...

(See Reference)

Pyramid

Actually Writing Formulae ...

(CP2 Topic)

Difficult part is to be fast ...

Dorothy's Pegging Game

Sky Full of Stars

Linear Recurrence

Basic Problem:

Let $a_1 = a_0 = 1$, $a_i = a_{i-1} + a_{i-2}$,

find $a_{1000000} \pmod{998244353}$

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Solution

$$\begin{pmatrix} a_i \\ a_{i-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{i-1} \\ a_{i-2} \end{pmatrix}$$

So $$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1}}_{\text{fast power}} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}.$$

Linear Recurrence

Complexlity : $O(k^3 \log n)$

(there are exotic ways to improve)

Other uses

Graph with path length exactly n.

So Easy !

Step 1: Convice yourself

$$S_n = \underbrace{(a+\sqrt{b})^n + (a-\sqrt{b})^n}_{\text{is an integer}}$$

Step 2: Find a recurrence between

(S_n, S_{n-1}, S_{n-2})

Summary

Pattern Finding

Come up with formulae

Linear Recurrence