Fundamentals of information transmission and coding (a.k.a. communication theory)

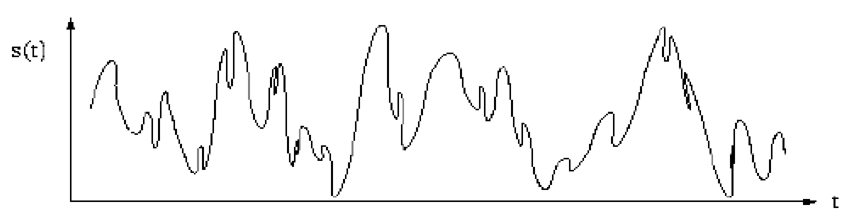
Signals and functions

Elementary operation of communication: send *signal* on medium from A to B.

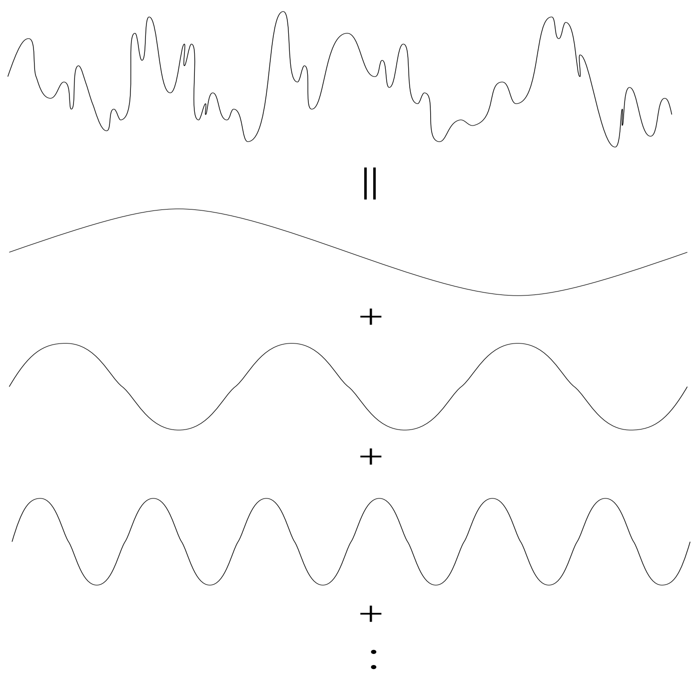
- media—copper wire, optical fiber, air/space, ...
- signals—voltage and currents, light pulses, radio waves, microwaves, ...

  $\rightarrow$ electromagnetic wave (let there be light!)

Signal can be viewed as a time-varying function $s(t)$.



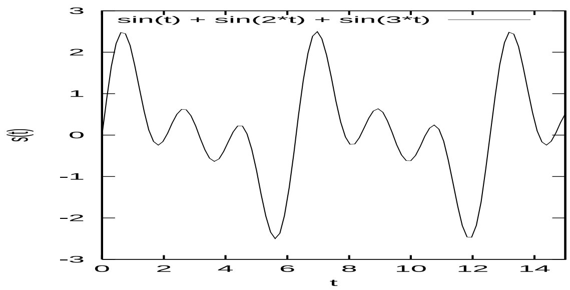
If $s(t)$ is "sufficiently nice" (Dirichlet conditions) then $s(t)$ can be represented as a linear combination of complex sinusoids:

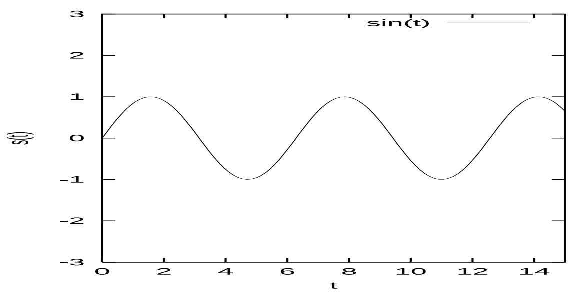


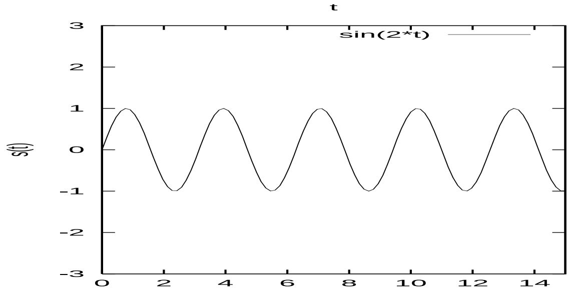
$\longrightarrow$ looks complicated

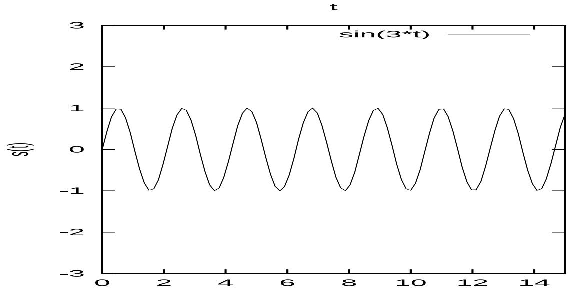
$\longrightarrow$ underneath: sum of simple building blocks

Simple example:



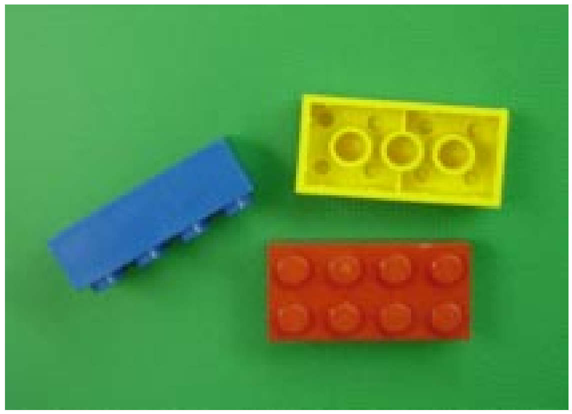


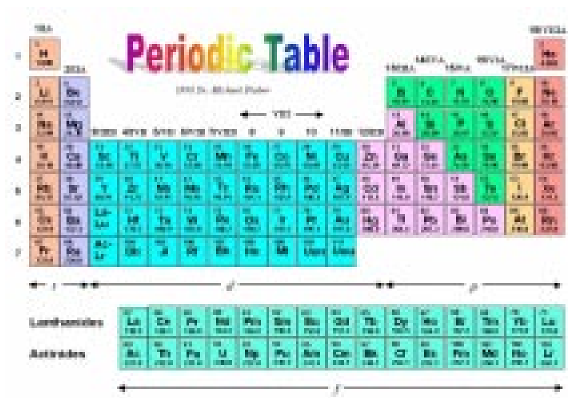




$\longrightarrow$ sinusoids form basis for other signals

Other examples (man-made & nature):





$\longrightarrow$ cells, atoms, strings, etc.

$\longrightarrow$ what's the connection to linear algebra?

Building blocks: analogous to "basis" in linear algebra

other elements (vectors) can be expressed as linear combinations of "elementary" elements (basis vectors)

$\longrightarrow$ bases like atoms

Ex.: in 3-D, $\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$ form a basis.

$\longrightarrow$ $(7,2,4) = 7 \cdot (1,0,0) + 2 \cdot (0,1,0) + 4 \cdot (0,0,1)$

$\longrightarrow$ coefficients: 7, 2, 4

$\longrightarrow$ "spectrum"

How many elements are there in a basis?

Vector spaces:

- finite dimensional

  $\longrightarrow$ linear algebra

  $\longrightarrow$ e.g., 7-dimensional: $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$

- infinite dimensional: signals

  $\longrightarrow$ $s(t)$: continuously varies with t

  $\longrightarrow$ like infinite number of bases

  $\longrightarrow$ bad news: cannot use linear algebra

  $\longrightarrow$ good news: concepts remain the same

  $\longrightarrow$ math: functional analysis

Given an arbitrary element in the vector space, how to find the coefficient of basis elements?

$\longrightarrow$ e.g., given $(7, 2, 4)$, coefficient of $(0, 1, 0)$?

But bases need not be $\{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$

$\longrightarrow$ $\{(2, 0, 0),\ (0, 4, 0),\ (0, 0, 5)\}$ is fine too

$\longrightarrow$ what's the spectrum of $(7, 2, 4)$?

$\longrightarrow$ is $\{(11, 0, 3),\ (2, 500, 7),\ (31, 44, 1)\}$ valid basis?

$\longrightarrow$ spectrum of $(7, 2, 4)$?

$\longrightarrow$ in general, to qualify as a basis ...

How to calculate coefficients of basis (spectrum)?

Given basis set $\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$

$\longrightarrow$ $(7,2,4) = 7 \cdot (1,0,0) + 2 \cdot (0,1,0) + 4 \cdot (0,0,1)$

$\longrightarrow$ spectrum: 7, 2, 4

$\longrightarrow$ "read off": cheating!

$\longrightarrow$ what's the general principle?

Recall "dot product" from linear algebra:

$$\longrightarrow \quad (x_1, x_2, x_3) \circ (y_1, y_2, y_3) = x_1 y_1 + x_2 y_2 + x_3 y_3$$

Ex.:

$$\longrightarrow \quad (1, 0, 0) \circ (1, 0, 0) = 1$$

$$\longrightarrow \quad (1, 0, 0) \circ (0, 1, 0) = 0$$

$$\longrightarrow \quad (1, 0, 0) \circ (0, 0, 1) = 0$$

What's special about basis set $\{(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)\}$?

How can it be used to calculate spectrum?

To compute spectrum of $(1,0,0)$ for $(7,2,4)$:

$\longrightarrow$ take dot product: $(7,2,4) \circ (1,0,0) = 7$

$\longrightarrow$ why does it work?

Since $(7,2,4) = 7 \cdot (1,0,0) + 2 \cdot (0,1,0) + 4 \cdot (0,0,1)$,

we have

$$
\begin{aligned}
(7,2,4) \circ (1,0,0) \\
&= [7 \cdot (1,0,0) + 2 \cdot (0,1,0) + 4 \cdot (0,0,1)] \circ (1,0,0) \\
&= 7 \cdot (1,0,0) \circ (1,0,0) \\
&\qquad + 2 \cdot (0,1,0) \circ (1,0,0) \\
&\qquad\qquad + 4 \cdot (0,0,1) \circ (1,0,0) \\
&= 7 \cdot 1 + 2 \cdot 0 + 4 \cdot 0 \\
&= 7
\end{aligned}
$$

$\longrightarrow$ light bulbs should go off!

$\longrightarrow$ super-powerful trick

Lastly: why do we care about spectra?

$\longrightarrow$ allows us to focus on what's important

Take $(7, 2, 4)$.

$\longrightarrow$ which building block is most important?

$\longrightarrow$ $(1, 0, 0)$ since it's multiplied by 7

$\longrightarrow$ then comes $(0, 1, 0)$, followed by $(0, 0, 1)$

From an approximation angle

$\longrightarrow$ $(7, 2, 4)$ kind of looks like $(7, 0, 0)$

$\longrightarrow$ $(7, 0, 4)$ is pretty close

$\longrightarrow$ $(7, 2, 4)$ is 100% accurate

In science & engineering:

$\longrightarrow$ rare luxury to have 100% accurate things

$\longrightarrow$ typically: must approximate

Ex.:

- compression: JPEG, MPEG are all lossy

  $\longrightarrow$ disk space forces us to approximate

  $\longrightarrow$ luckily human eye (or is it the brain?) does the same

- caching: memory hierarchy

  $\longrightarrow$ "cache $\mapsto$ RAM $\mapsto$ disk"

  $\longrightarrow$ cache contains approximation of memory

  $\longrightarrow$ memory contains approximation of disk

  $\longrightarrow$ luckily it works

  $\longrightarrow$ programs obey locality-of-reference

- etc.

For signals that represent bits in networking

$\longrightarrow$ take the same attitude

A complicated looking signal $s(t)$ may be replaced by a much simpler looking approximation $s'(t)$

$\longrightarrow$ then work with $s'(t)$

$\longrightarrow$ keep life simple

$\longrightarrow$ don't sweat the little things (except when coding)

$\longrightarrow$ Amdahl's law

$\longrightarrow$ then there are additional benefits . . .

On to signals: $s(t)$ can be viewed as

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega t} d\omega,$$

⟶ signal $s(t)$ is a linear combination of the $e^{i\omega t}$'s

⟶ recall: $e^{i\omega t} = \cos \omega t + i \sin \omega t$

⟶ building block: sine curve

⟶ basically: weighted sum of sine curves

⟶ fancy name: Fourier expansion

⟶ $S(\omega)$: coefficient/weight of basis elements

Frequency ω: cycles per second (Hz)

⟶ $\omega = 1/T$ where T is the period

To simplify, we need to know which sine curves contribute most

⟶ need to know $S(\omega)$

Simple rule to compute $S(\omega)$:

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt.$$

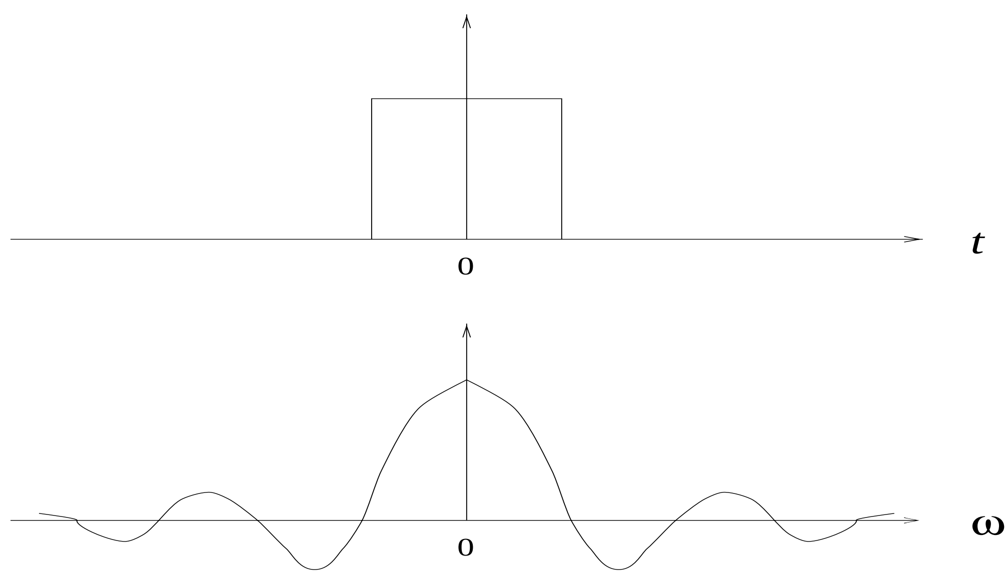
$\longrightarrow$ does it look like a "dot product"?

$\longrightarrow$ to simplify: throw out all sines with "small" $S(\omega)$

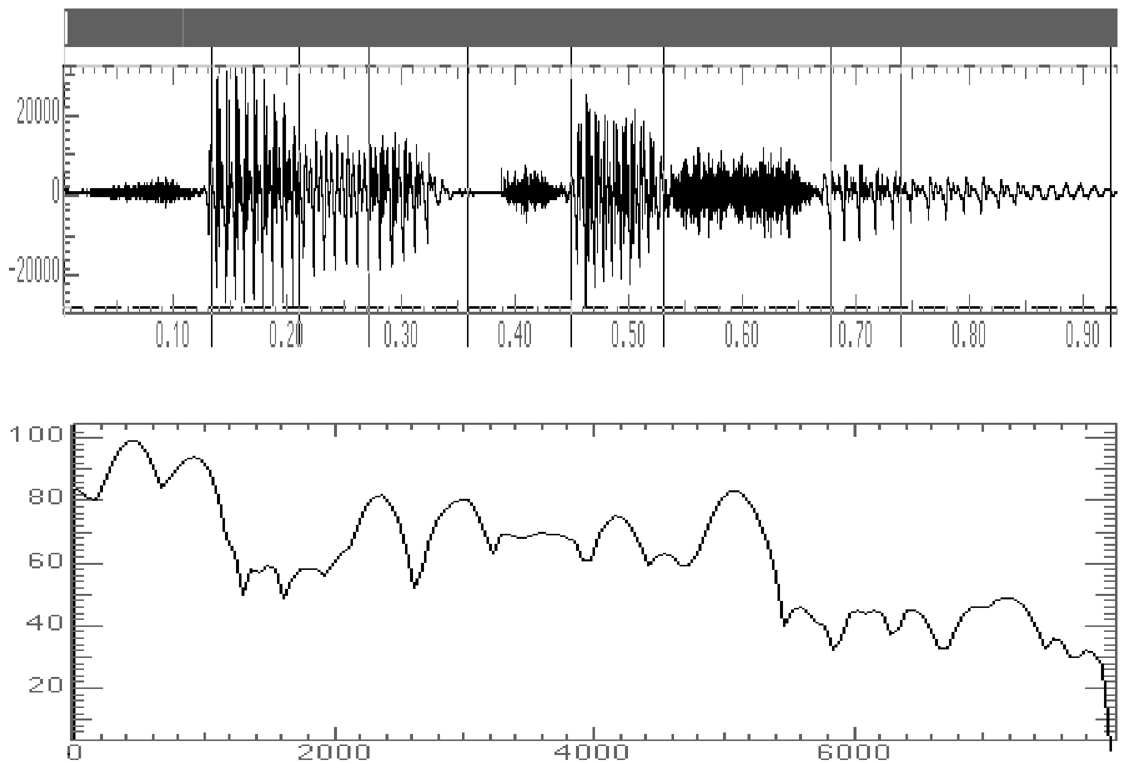
$\longrightarrow$ how small is small?

Example: square wave

$\longrightarrow$ $s(t)$ and $S(\omega)$ profiles



Example: audio (e.g., speech) signal



Source: Dept. of Linguistics and Phonetics, Lund University

Random signal (i.e., white noise) has "flat-looking" spectrum.

$\longrightarrow$ unbounded bandwidth

$\longrightarrow$ cannot compress

$\longrightarrow$ what about "snow" on TV screen?

Luckily, most "interesting" functions arising in practice are "special":

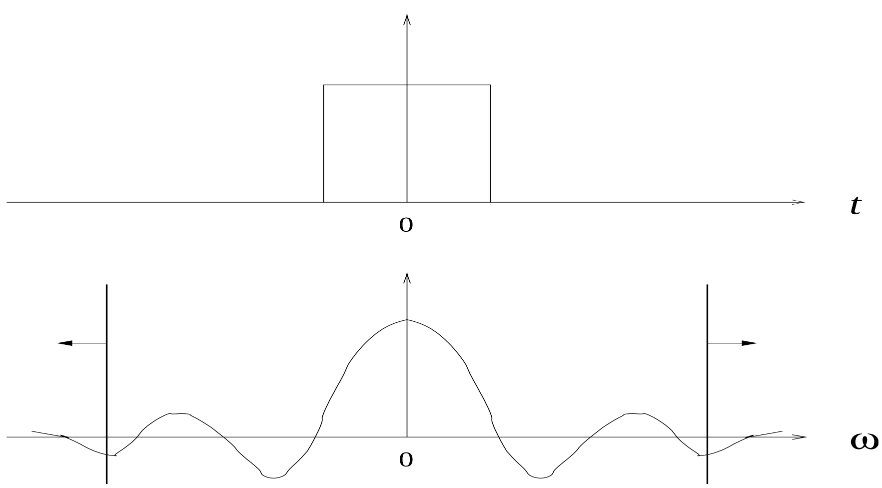
$\longrightarrow$ bandlimited

$\longrightarrow$ i.e., $S(\omega) = 0$ for $|\omega|$ sufficiently large

$\longrightarrow$ when $S(\omega) \approx 0$, can treat as $S(\omega) = 0$

$\longrightarrow$ let's approximate!

$\longrightarrow$ e.g., square wave: cut the tails off $S(\omega)$



Ex.: human auditory system

$\longrightarrow$ 20 Hz–20 kHz

$\longrightarrow$ speech is intelligible at 300 Hz–3300 Hz

$\longrightarrow$ broadcast quality audio; CD quality audio

Telephone systems: engineered to exploit this property

$\longrightarrow$ bandwidth 3000 Hz

$\longrightarrow$ copper medium: various grades

$\longrightarrow$ physical media: damages traveling signals

$\longrightarrow$ no problem transmitting 3000 Hz signals

$\longrightarrow$ if transmit 2 GHz signal: corruption large

For communication:

Both absolute frequency and band**width** are relevant.

$\longrightarrow$ baseband vs. broadband

$\longrightarrow$ high-speed $\Leftrightarrow$ broadband

Manipulate shape of different frequency sinusoids to **simultaneously** carry information (i.e., bits).

$\longrightarrow$ multi-lane highway analogy

$\longrightarrow$ different lane $\Leftrightarrow$ different frequency

$\longrightarrow$ can craft signals to our liking

$\longrightarrow$ engineering application important for communication