Randomized Rounding

- Similar to the deterministic rounding, except that optimal fractional solution is rounded randomly according to some probability distribution.

- Goal is to obtain a good approximation ratio with high probability

- Independently run the rounding multiple times to improve the above probability.

$O(\log n)$-approximation for Set Cover

Problem: Given sets $S_1, S_2, \ldots, S_m$ of $\{1, 2, \ldots, n\}$, find smallest subset C s.t. $C \cap S_k \neq \phi$ for each k.

LP relaxation:

$$\min \sum_i x_i$$
$$\sum_{i \in S_k} x_i \geq 1 \ \forall k$$
$$0 \leq x_i \leq 1.$$

Randomized Rounding:

$$y_i = \text{rounded}(x_i)$$
$$y_i = 1 \text{ with prob.}(x_i) \text{ and}$$
$$= 0 \quad \text{"} \quad \text{prob. } (1 - x_i).$$

$\forall k$, $\Pr[S_k \text{ gets covered}] = 1-(1-x_{k1})\cdots(1-x_{k\ell})$

where $S_k = \{k_1, k_2, \ldots k_\ell\}$.

We have

$$(1-x_{k_1})\cdots(1-x_{k_\ell}) \leq \left(\frac{\ell-(x_{k_1}+x_{k_2}+\ldots x_{k_\ell})}{\ell}\right)^\ell$$

$$\leq \left(1-\frac{1}{\ell}\right)^\ell$$

$$\leq \frac{1}{e}$$

So,

$$\Pr[S_k \text{ gets covered}] \geq 1-\frac{1}{e}$$

- Now repeat the randomized rounding t times and take the union of all the sets produced.
- This means we round x_i to 0 with probability $(1-x_i)^t$, that x_i is rounded to 0 if it is rounded to 0 in all t rounds.

Prob.[s_k remains uncovered] $\leq (\frac{1}{e})^t$ after t rounds.

Prob.[any of s_k remains uncovered]

$$\leq m \cdot (\tfrac{1}{e})^t.$$

If we take $t = \log_e m + 1$, we see

Prob.[any of s_k remains uncovered]

$$\leq \frac{1}{em} \cdot m = \frac{1}{e} < 1$$

- Size bound

E[size of C after one round]

$$= \sum_i x_i$$

$$= V(LP) \leq OPT$$

E[Size of C after t rounds]

$$\leq t \cdot OPT$$

By Markov's inequality,

$$\text{Prob.}\left[\text{Size of } C > \rho.\text{OPT}\right]$$

$$\leq \frac{E[\text{Size of } C]}{\rho.\ \text{OPT}}$$

$$\leq \frac{t.\ \text{OPT}}{\rho.\ \text{OPT}} = \frac{t}{\rho}$$

$$\text{Prob.}\left[\text{All } S_k \text{ covered and Size } C \leq \rho.\text{OPT}\right]$$

$$\geq \left(1 - m\left(\tfrac{1}{e}\right)^t\right)\left(1 - \frac{t}{\rho}\right)$$

$$\geq 1 - m\left(\tfrac{1}{e}\right)^t - \frac{t}{\rho}$$

Put $t = \Theta(\log m)$ and $\rho = 4t$ so that

$$1 - m\left(\tfrac{1}{e}\right)^t - \frac{t}{\rho} \geq \frac{1}{2}.$$

Thm. This algorithm produces a solution with an $O(\log m)$ approximation factor with probability at least $\frac{1}{2}$.