Persistence and Beyond

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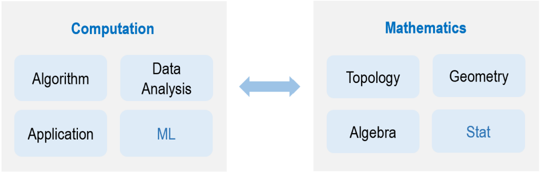
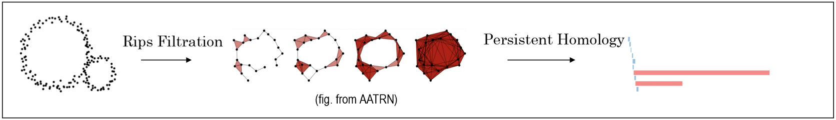
April 16, 2025

Persistent Homology

Standard Persistence Homology Pipeline

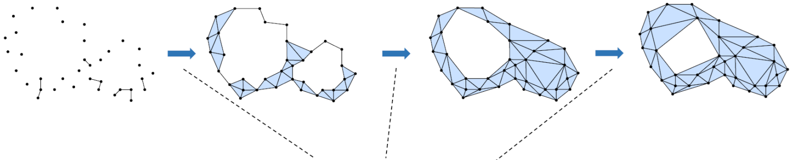
Topological data analysis (TDA) • Computational topology





(fig. from AATRN)

Persistence



Expand each arrow into a sequence of additions of a single simplex

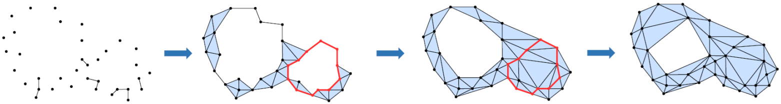
Filtration: a sequence of additions of a single simplex

$$\mathcal{F} : \varnothing = K_0 \xhookrightarrow{\sigma_1} K_1 \xhookrightarrow{\sigma_2} \cdots \xhookrightarrow{\sigma_{m-1}} K_{m-1} \xhookrightarrow{\sigma_m} K_m$$

Persistent homology

Input: $\mathcal{F} : \varnothing = K_0 \xhookrightarrow{\sigma_1} K_1 \xhookrightarrow{\sigma_2} \cdots \xhookrightarrow{\sigma_{m-1}} K_{m-1} \xhookrightarrow{\sigma_m} K_m$

An interval $[b, d]$: starts and ends with **indices in the filtration**



Persistent homology

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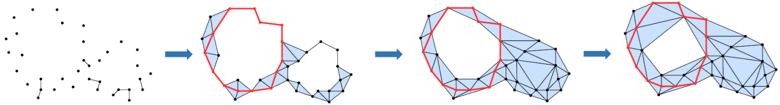
An interval $[b, d]$: starts and ends with **indices in the filtration**



Standard persistence

Standard filtration:

$$\mathcal{F} : K_0 \xhookrightarrow{\sigma_0} K_1 \xhookrightarrow{\sigma_1} \cdots \xhookrightarrow{\sigma_{m-2}} K_{m-1} \xhookrightarrow{\sigma_{m-1}} K_m$$

$$\Downarrow$$

Induced module:

$$\mathsf{H}_p(\mathcal{F}) : \mathsf{H}_p(K_0) \to \mathsf{H}_p(K_1) \to \cdots \to \mathsf{H}_p(K_{m-1}) \to \mathsf{H}_p(K_m)$$

$$\Downarrow$$

Interval decomposition: [**Gabriel 72**]

$$\mathsf{H}_p(\mathcal{F}) = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{I}^{[b_\alpha, d_\alpha]}$$

$$\Downarrow$$

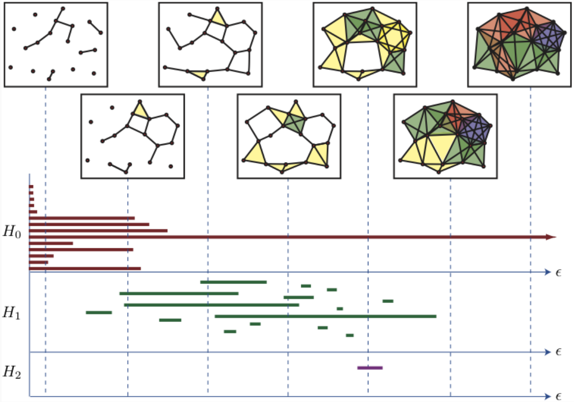
p-th persistence barcode:

$$\mathsf{Pers}_p(\mathcal{F}) = \{[b_\alpha, d_\alpha] \mid \alpha \in \mathcal{A}\}$$

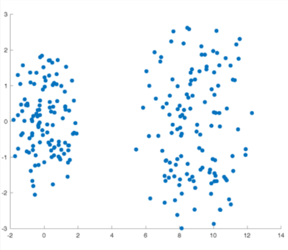
Peter Gabriel. Unzerlegbare Darstellungen I. Manuscripta Mathematica, 6(1):71–103, 1972.

Bar Codes

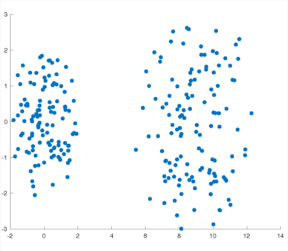
- birth-death and bar codes

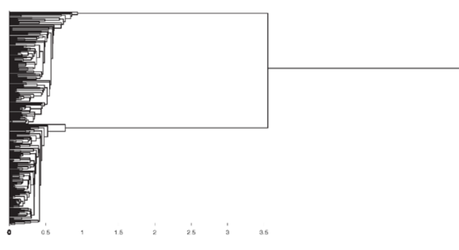


Clustering and Barcode



Clustering and Barcode





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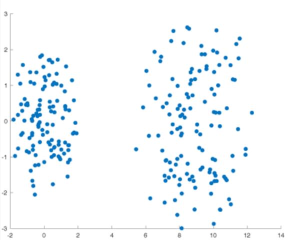

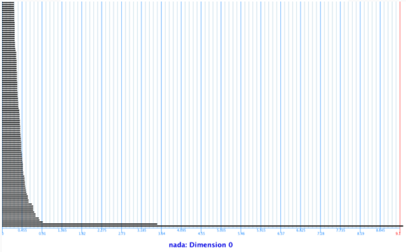

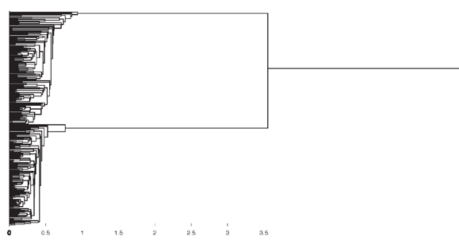

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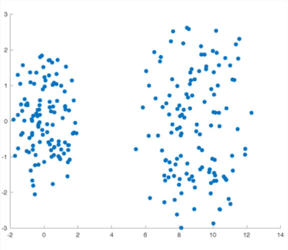
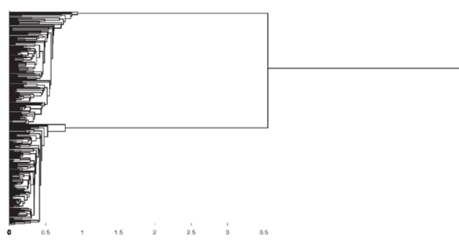
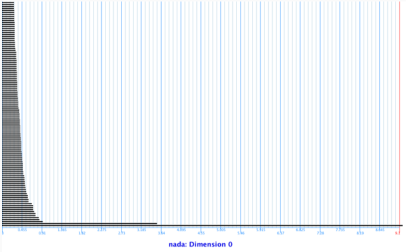
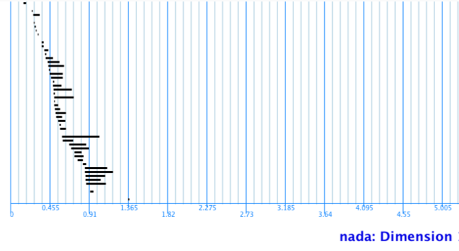
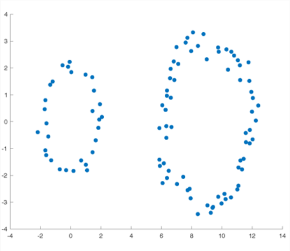








Holes and Barcode



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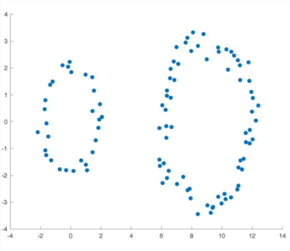

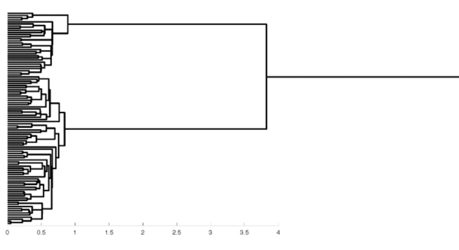

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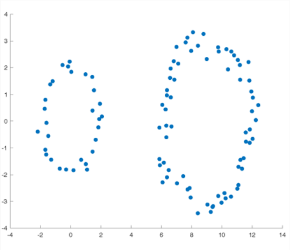

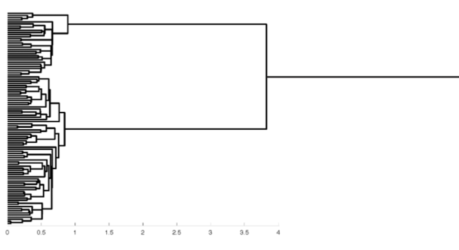

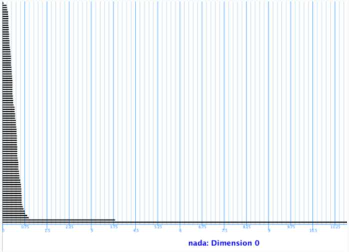

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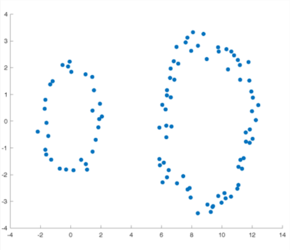

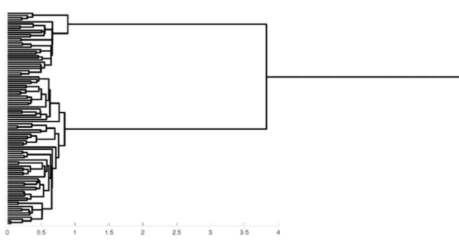

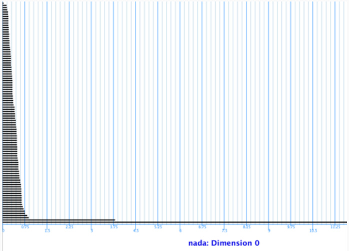

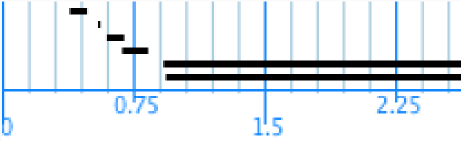

Zigzag filtration

Zigzag filtration:

$$\mathcal{F} : K_0 \xleftrightarrow{\sigma_0} K_1 \xleftrightarrow{\sigma_1} \cdots \xleftrightarrow{\sigma_{m-2}} K_{m-1} \xleftrightarrow{\sigma_{m-1}} K_m$$

$$K_i \xhookrightarrow{\sigma_i} K_{i+1} \text{ or } K_i \xhookleftarrow{\sigma_i} K_{i+1}$$

Gunnar Carlsson and Vin de Silva. **Zigzag persistence**.
Foundations of Computational Mathematics, 10(4):367–405, 2010.

Zigzag persistence

Zigzag filtration:

$$\mathcal{F} : K_0 \overset{\sigma_0}{\longleftrightarrow} K_1 \overset{\sigma_1}{\longleftrightarrow} \cdots \overset{\sigma_{m-2}}{\longleftrightarrow} K_{m-1} \overset{\sigma_{m-1}}{\longleftrightarrow} K_m$$

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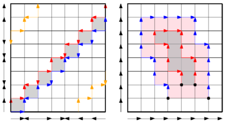
Peter Gabriel. Unzerlegbare Darstellungen I. Manuscripta Mathematica, 6(1):71–103, 1972.

Zigzag persistence

- In time varying setting: functions, point cloud, network, vector field

  - G. Carlsson, V. de Silva, and Dmitriy Morozov. Zigzag persistent homology and real-valued functions. 2009.
  - W. Kim and F. Me'moli. Spaiotemporal persistent homology of dynamic metric spaces, Discrete & Comput. Geom. 2020, pages 1—45.
  - A. Myers, D. Munoz, F. Khasawneh, E. Munch. Temporal network analysis using zigzag persistence. Springer Nature, 2023.
  - T. Dey, M. Lipinsky, M. Mrozek, R. Slechta. Tracking dynamical features via continuation and persistence. 38th Annu. Symposium on Comput. Geometry, 2022.

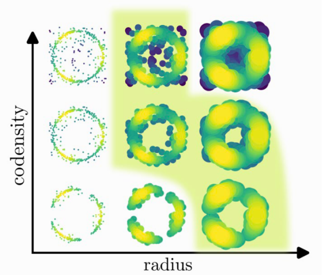
- In multiparameter persistence (computing generalized rank, [DKM22])



2-parameter Persistence

2-parameter Persistence

- In the previous setting, we had a 1-parameter filtration. What if we have a 2-parameter filtration?



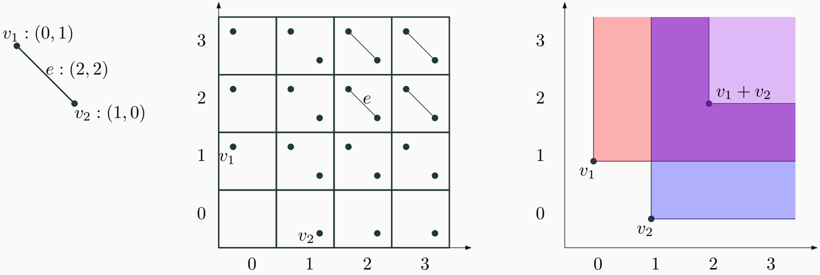
$$
\begin{array}{ccccccc}
X_{\alpha_0,\beta_n} & \subseteq & X_{\alpha_1,\beta_n} & \subseteq & \dots & \subseteq & X_{\alpha_n,\beta_n} \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\subseteq$} \\
\vdots & & \vdots & & & & \vdots \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\subseteq$} \\
X_{\alpha_0,\beta_1} & \subseteq & X_{\alpha_1,\beta_1} & \subseteq & \dots & \subseteq & X_{\alpha_n,\beta_1} \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\subseteq$} \\
X_{\alpha_0,\beta_0} & \subseteq & X_{\alpha_1,\beta_0} & \subseteq & \dots & \subseteq & X_{\alpha_n,\beta_0}
\end{array}
$$

2-parameter Filtration

$$\begin{array}{ccccccc}
K_{\alpha_0,\beta_n} & \subseteq & K_{\alpha_1,\beta_n} & \subseteq & \cdots & \subseteq & K_{\alpha_n,\beta_n} \\
\cup\!| & & \cup\!| & & & & \cup\!| \\
\vdots & & \vdots & & & & \vdots \\
\cup\!| & & \cup\!| & & & & \cup\!| \\
K_{\alpha_0,\beta_1} & \subseteq & K_{\alpha_1,\beta_1} & \subseteq & \cdots & \subseteq & K_{\alpha_n,\beta_1} \\
\cup\!| & & \cup\!| & & & & \cup\!| \\
K_{\alpha_0,\beta_0} & \subseteq & K_{\alpha_1,\beta_0} & \subseteq & \cdots & \subseteq & K_{\alpha_n,\beta_0}
\end{array}$$

2-parameter Persistence

Bi-filtration and induced 2-parameter persistence module.



2-parameter persistence in Machine Learning

Vectorization of 2-parameter persistence

- Multiparameter persistence landscapes, (Vipond, 2020)
- Multiparameter persistence images, (Corbet et al. 2019)
- Multiparameter persistence kernel, (Carriére et al. 2020)
- Vectorization of signed barcodes (Loiseaux et al. 2023)

GRIL: Generalized Rank Invariant Landscapes

[Xin,Mukherjee,Samaga,D. 23] GRIL, TAG-ML 2023: 313-333

Generalized Rank

Definition

Let $M\colon P \to \mathbf{vec}$ be a persistence module, P be finite and connected poset. Then, the generalized rank of M over a subposet $I \subseteq P$ is:

$$\mathrm{rk}^M(I) := rank\left(\varprojlim M|_I \to \varinjlim M|_I\right).$$

This definition is due to (Kim and Mémoli, 2021).

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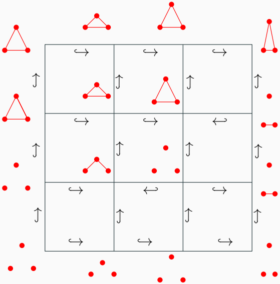
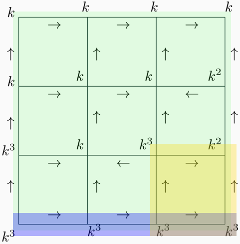
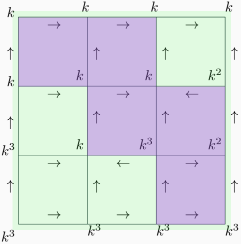
This definition is due to (Kim and Mémoli, 2021).
If I is a rectangle, $rk^M(I)$ is the usual rank of the map on the diagonal.

[Kim,Mémoli 21]: Generalized persistence diagrams for persistence modules over posets. Journal of Applied and Computational Topology, 5(4):533–581.

A simple example

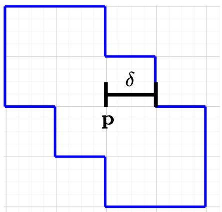






Worms

- $\boxed{\mathbf{p}}_\delta := \{\mathbf{w} : \|\mathbf{p} - \mathbf{w}\|_\infty \leq \delta\}$ be the p-centered δ-square.
- a *worm* $\boxed{\mathbf{p}}_\delta^2$ is the union of $\boxed{\mathbf{p}}_\delta$ with two δ-squares $\boxed{\mathbf{q}}_\delta$ with $\mathbf{q} = \mathbf{p} \pm \delta \cdot (1, -1)$.

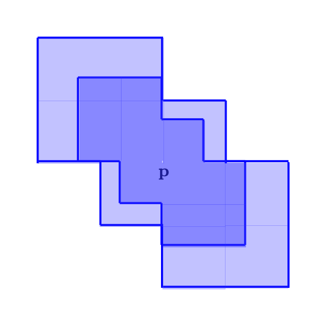


Generalized Rank Invariant Landscapes

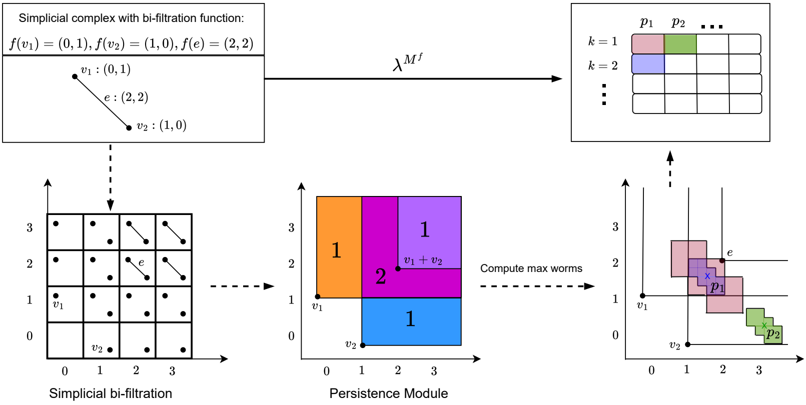
Definition

For a 2-parameter persistence module M, Generalized Rank Invariant Landscape ($\textsc{Gril}$) is a function $\lambda^M\colon \mathbb{Z}^2 \times \mathbb{N} \to \mathbb{N}$ given by

$$\lambda^M(\mathbf{p}, k) := \sup\left\{\delta \geq 0 : rk^M\left(\boxed{\mathbf{p}}^2_\delta\right) \geq k\right\}$$



Generalized Rank Invariant Landscapes



Properties of GRIL

Proposition

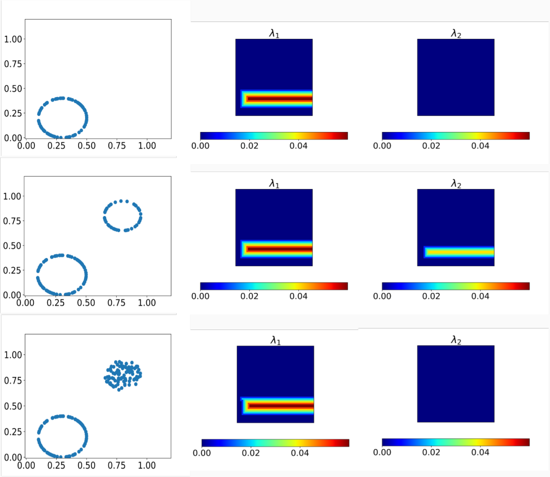
GRIL *is stable with respect to the input bi-filtration functions, i.e., given* $f, g \colon X \to \mathbb{R}^2$, *then*

$$||\lambda^{M_f} - \lambda^{M_g}||_\infty \leq ||f - g||_\infty$$

Proposition

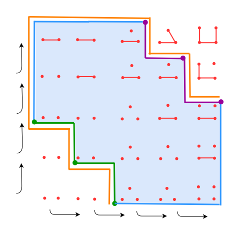
GRIL *is differentiable almost everywhere.*

GRIL as a topological discriminator



Computing Generalized Rank using Zigzag

- Generalized Rank for a worm $I = \#$ full bars in zigzag module along its boundary cap ∂I (Dey et al., 2024).
- Recent work on fast computation of zigzag persistence (Dey and Hou, 2022):
  https://github.com/TDA-Jyamiti/fzz



[D. Kim Mémoli 24] Computing Generalized Rank Invariant for 2-Parameter Persistence Modules via Zigzag Persistence and Its Applications. Discret. Comput. Geom. 71(1): 67-94 (2024)

[D. Hou 22] Fast Computation of Zigzag Persistence. ESA 2022: 43:1-43:15

Experimental Results

Experimental Results

- Gril on Proteins, Dhfr, Imdb-Binary, Cox2, Mutag, which are benchmark graph datasets.
- The aim is graph classification. Above datasets have two classes, thus it is a binary classification task.
- Heat-Kernel Signature and Ricci Curvature are chosen as the two functions to form a bi-filtration on the graphs.
- Computed the Gril values for $k \in \{1, 2, 3, 4, 5\}$.
- Sample **p** from different uniform sub-grids.

Experimental Results

Dataset	MP-I	MP-K	MP-L	P	Gril
Proteins	67.3 ± 3.5	67.5 ± 3.1	65.8 ± 3.3	65.4 ± 2.7	70.9 ± 3.1
Dhfr	80.2 ± 2.3	81.7 ± 1.9	79.5 ± 2.3	70.9 ± 3.1	77.6 ± 2.5
Cox2	77.9 ± 2.7	79.9 ± 1.8	79.0 ± 3.3	76.0 ± 4.1	79.8 ± 2.9
Mutag	85.6 ± 7.3	86.2 ± 2.6	85.7 ± 2.5	79.2 ± 7.7	87.8 ± 4.2
Imdb-Binary	71.1 ± 2.1	68.2 ± 1.2	71.2 ± 2.0	54.0 ± 1.9	65.2 ± 2.6

- We used a 50×50 grid.
- We used XGBoost classifier.
- The accuracies are averaged over 5 train/test splits of the datasets obtained with 5 stratified folds.

Experimental Results 2

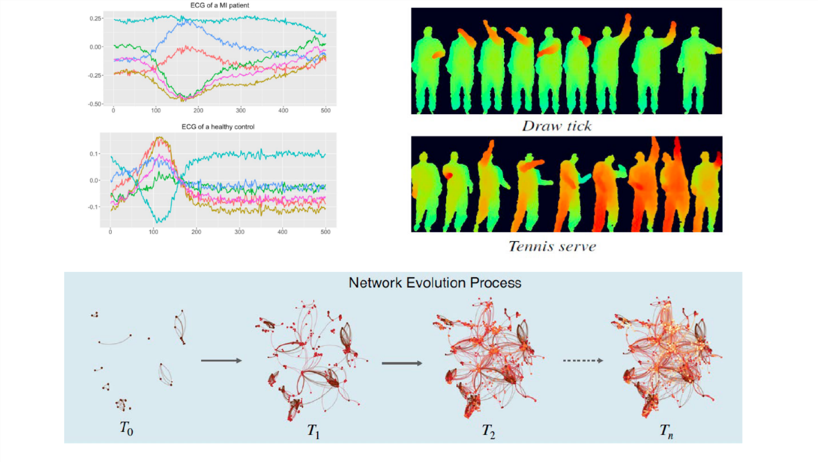
Model	Proteins	Dhfr	Cox2	Mutag	Imdb-Binary
GCN	71.15 ± 2.31	78.70 ± 2.35	78.80 ± 2.13	88.26 ± 3.70	73.1 ± 2.20
GCN + GRIL	74.21 ± 2.08	75.66 ± 3.08	80.30 ± 1.57	88.80 ± 3.60	72.6 ± 1.46
GAT	67.66 ± 3.92	77.78 ± 4.50	79.45 ± 3.68	86.69 ± 6.36	74.90 ± 2.98
GAT + GRIL	71.60 ± 3.92	79.64 ± 6.29	80.52 ± 3.30	84.03 ± 7.85	71.60 ± 3.04
GIN	69.09 ± 3.77	79.77 ± 6.72	78.80 ± 4.88	83.97 ± 6.04	73.7 ± 3.34
GIN + GRIL	71.87 ± 3.22	78.46 ± 5.80	79.22 ± 4.89	89.32 ± 4.81	74.2 ± 2.82

- We used a 50×50 grid.
- Gril features are augmented with standard GNNs.

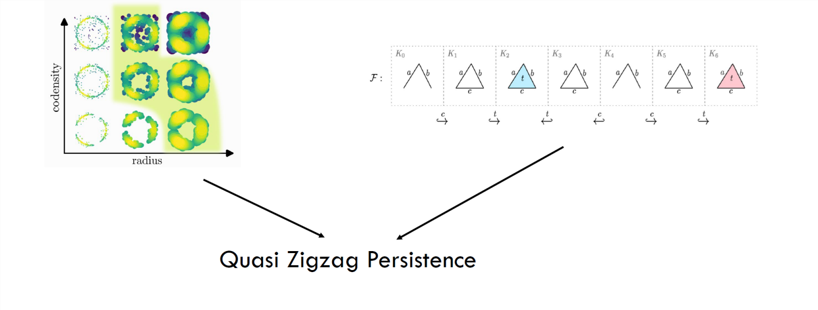
ZZ-GRIL

[D.,Samaga 25]: Quasi Zigzag Persistence

Time-varying data



Quasi Zigzag Persistence



[Dey,Samaga 25]: Quasi Zigzag Persistence: A Topological Framework for Analyzing Time-Varying Data. ArXiv: abs/2502.16049 (2025)

Quasi zigzag persistence

$$
\begin{array}{ccccccccc}
K_{0,\beta_n} & \subseteq & K_{1,\beta_n} & \supseteq & K_{2,\beta_n} & \subseteq & \dots & \supseteq & K_{n,\beta_n} \\
\cup| & & \cup| & & \cup| & & & & \cup| \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
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\end{array}
$$

ZZ-GRIL

Definition (Quasi zigzag poset and persistence module)

- QZ poset: subposet of $\mathbb{ZZ} \times \mathbb{Z}$
- A *quasi zigzag persistence module* $M : P \to \mathbf{vec}$, P is a QZ poset.

ZZ-GRIL

Definition (Quasi zigzag poset and persistence module)

- QZ poset: subposet of $\mathbb{ZZ} \times \mathbb{Z}$
- A *quasi zigzag persistence module* $M : P \to \mathbf{vec}$, P is a QZ poset.

Definition (ZZ-GRIL)

- M a QZ module.
- *ZigZag Generalized Rank Invariant Landscape* is a function $\lambda^M : \mathbb{ZZ} \times \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$

$$\lambda^M(\mathbf{p}, k) := \sup \left\{ \delta \geq 0 \colon \mathrm{rk}^M \left(\boxed{\mathbf{p}}_{\delta}^{2} \right) \geq k \right\},$$

where $\mathbf{p} \in \mathbb{ZZ} \times \mathbb{Z}$.

Properties of ZZ-GRIL

Theorem

Let M and N be two QZ modules. Let the $\textsc{Zz-Gril}$ of M and N over $\boxed{\mathbf{p}}_{\delta}^{2}$ be λ^M and λ^N respectively. Then,

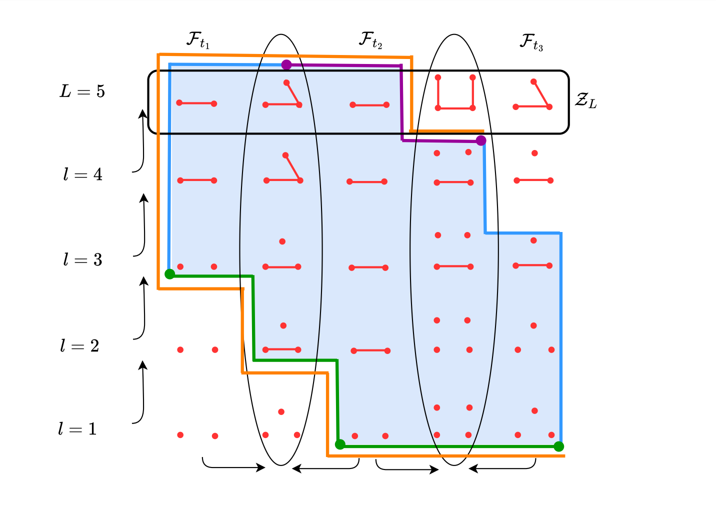
$$|\lambda^M(\mathbf{p},k) - \lambda^N(\mathbf{p},k)| = d_{\mathcal{E}}^{\mathcal{L}}(M,N) \leq d_{\mathcal{I}}(M,N).$$

Corollary (Stability)

Let M and N be two QZ modules. Then,

$$||\lambda^M - \lambda^N||_\infty \leq d_{\mathcal{I}}(M,N).$$

Algorithm to compute ZZ-GRIL



[DKM24] result cannot be applied to QZ poset.

Computing ZZ-GRIL

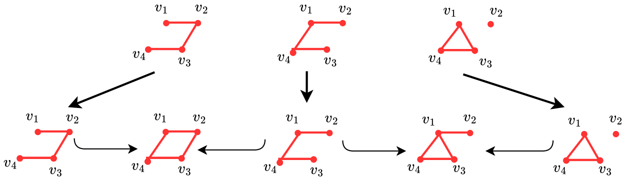
- Concepts of initial and terminal functors help in extending the result in [DKM24] to QZ modules.
- Proof in a recent preprint to be available on arXiv soon.

Theorem

Let M be a quasi zigzag persistence module and I be a finite interval in the corresponding quasi zigzag poset. Then, $\mathrm{rk}^M(I) = \mathrm{rk}^M(\partial I)$.

Building quasi zigzag bi-filtration

A typical input

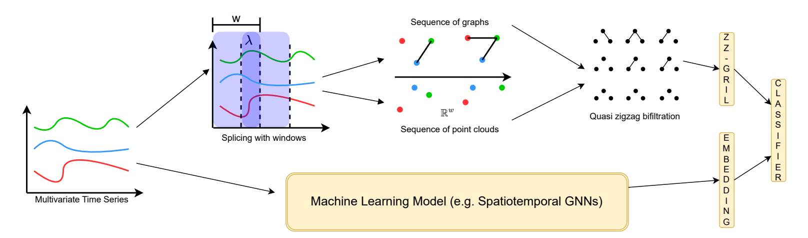


Experimental Results

Experiments

- We perform experiments on Multivariate Time-Series Classification and Sleep-Stage Classification.
- We use the UEA multivariate time-series datasets (Bagnall et al., 2018) for multivariate time-series classification and the ISRUC-S3 dataset (Khalighi et al., 2016) for sleep-stage classification.
- We augment Zz-Gril features to an existing machine learning architecture and compare the performance.
- For the experiments, we use $k = 1, 2, 3$ and compute Zz-Gril at 36 sampled center points.

Experiments



Experimental Results

Here, we report the performance of Zz-Gril augmented to STDP-GCN (Zhao et al., 2023) on sleep-stage classification, which is a 5-way classification task.

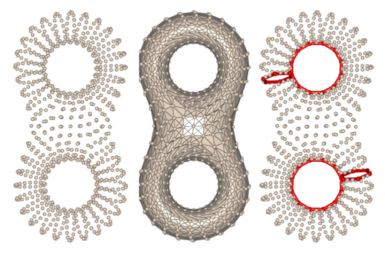
Methods	Accuracy	F-1 Overall	F-1 Wake	F-1 N-1	F-1 N2	F-1 N3	F-1 REM
SVM (Alickovic and Subasi, 2018)	73.3	72.1	86.8	52.3	69.9	78.6	73.1
RF (Memar and Faradji, 2017)	72.9	70.8	85.8	47.3	70.4	80.9	69.9
MLP+LSTM (Dong et al., 2017)	77.9	75.8	86.0	46.9	76.0	87.5	82.8
CNN+BiLSTM (Supratak et al., 2017)	78.8	77.9	88.7	60.2	74.6	85.8	80.2
CNN (Chambon et al., 2018)	78.1	76.8	87.0	55.0	76.0	85.1	80.9
ARNN+RNN (Phan et al., 2019)	78.9	76.3	83.6	43.9	79.3	87.9	86.7
STGCN (Jia et al., 2020)	79.9	78.7	87.8	57.4	77.6	86.4	84.1
MSTGCN (Jia et al., 2021)	82.1	80.8	**89.4**	59.6	80.6	**89.0**	85.6
STDP-GCN (Zhao et al., 2023)	82.6	81.0	83.5	**62.9**	83.1	86.0	90.6
STDP-GCN + Zz-Gril	**83.8**	**81.1**	88.6	58.1	**85.4**	82.7	**90.9**

Experimental Results

Here, we report the performance of Zz-Gril augmented to TodyNet (Liu et al., 2024) on various multivariate time-series datasets.

Dataset/Methods	ED-1NN	DTW-1NN-I	DTW-1NN-D	MLSTM-FCN	ShapeNet	WEASEL+MUSE	TapNet	OS-CNN	MOS-CNN	TodyNet	TodyNet+Zz-Gril
FingerMovements	0.550	0.520	0.530	0.580	0.589	0.490	0.530	0.568	0.568	0.570	**0.660**
Heartbeat	0.620	0.659	0.717	0.663	**0.756**	0.727	0.751	0.489	0.604	**0.756**	**0.756**
MotorImagery	0.510	0.390	0.500	0.510	0.610	0.500	0.590	0.535	0.515	0.640	**0.660**
NATOPS	0.860	0.850	0.883	0.889	0.883	0.870	0.939	0.968	0.951	**0.972**	0.961
SelfRegulationSCP2	0.483	0.533	0.539	0.472	0.578	0.460	0.550	0.532	0.510	0.550	**0.600**

Dataset	**TodyNet + Zz-Gril point clouds**	**TodyNet + Zz-Gril graphs**
FingerMovements	0.660	0.680
NATOPS	0.961	0.945
SelfRegulationSCP2	0.600	0.594



CGTDA group at Purdue University

Links for softwares: https://github.com/TDA-Jyamiti/gril
https://github.com/TDA-Jyamiti/zzgril

L-R: S. Mukherjee, S. Samaga, C. Xin, T. Hou

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