

**Note:** Use L<sup>A</sup>T<sub>E</sub>X or Word to typeset your solutions, or write them legibly by hand and submit the scan. You can use the source code of this file as a template.

**Problem 1 (One-way infinite tape).** A one-way infinite tape has a leftmost cell, and only goes infinitely to the right direction. A Turing machine with a one-way infinite tape is initialized with the input on the leftmost block of cells. Every time a transition tries to move the head to the left of the leftmost cell, the head will instead stay.

We will prove that this is equivalent to a normal Turing machine with a two-way infinite tape.

1. (10 pts). Let  $M$  be a Turing machine with a single one-way infinite tape. Formally describe a normal single-tape Turing machine  $M'$  such that  $M'(w) = M(w)$  for every input  $w$ .
2. (10 pts). Let  $M$  be a normal single-tape Turing machine. Formally describe a Turing machine with a single one-way infinite tape  $M'$  such that  $M'(w) = M(w)$  for every input  $w$ .

**Problem 2 (Prove recognizability).** Prove the following languages are recognizable.

1. (5 pts).  $R_{\text{TM}} = \{(M, w) \mid \text{Turing machine } M \text{ rejects } w\}$ .
2. (5 pts).  $A_{\text{NTM}} = \{(M, w) \mid \text{nondeterministic Turing machine } M \text{ accepts } w\}$ .
3. (5 pts).  $\overline{\text{EMPTY}_{\text{TM}}} = \{M \mid M \text{ is a Turing machine and } L(M) \neq \emptyset\}$ .

**Problem 3 (Closure properties with Turing machines).** Prove the following closure or non-closure properties on decidable and recognizable languages.

1. (5 pts). Prove that  $R$  is closed under Kleene star.
2. (10 pts). Prove that  $RE$  is closed under concatenation.
3. (10 pts). Fixed a symbol  $s \in \Sigma$ . For every string  $w \in L$ , define its  $s$ -deletion  $w_{\ominus s}$  as the remaining string after removing all  $s$  in  $w$ . That is, if  $w = w_1 \cdots w_n$ , then

$$w_{\ominus s} = y_1 \cdots y_n, \quad y_i = \begin{cases} w_i & \text{if } w_i \neq s; \\ \varepsilon & \text{if } w_i = s. \end{cases}$$

For a language  $L \subseteq \Sigma^*$ , define the  $s$ -deletion as:

$$L_{\ominus s} = \{w_{\ominus s} \mid w \in L\}.$$

Prove that RE is closed under deletion.

4. **(10 pts)**. Show that R is not closed under deletion.

*Hint.* You can construct a counterexample for this question with an alphabet of size 3.

**Problem 4 (Ordered enumeration).** A language  $L$  is orderly enumerable, if there exists a enumerator Turing machine that enumerates strings in  $L$  in an increasing order by length.

1. **(20 pts)**. Prove that a language is orderly enumerable if and only if it is decidable.
2. **(10 pts)**. Prove that every infinite recognizable language contains a infinite subset that is decidable.

**Problem 5 (Read-only Turing machine).** **(Bonus, 20 pts)**. A Turing machine  $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$  is read-only, if it never rewrite the cells. That is, for every  $q \in Q$  and  $s \in \Gamma$ , it holds that

$$\delta(q, s) = (q', s, D)$$

for some  $q' \in Q$  and  $D \in \{\leftarrow, \rightarrow\}$ . Prove that a language can be recognized by a read-only Turing machine if and only if it is regular.

*Hint.* It could be helpful to consider the communication model, where Alice has the prefix  $w_1 \cdots w_i$  and Bob has the suffix  $w_{i+1} \cdots w_n$ . What information does Alice need to tell Bob so that Bob can simulate the execution of the read-only Turing machine?