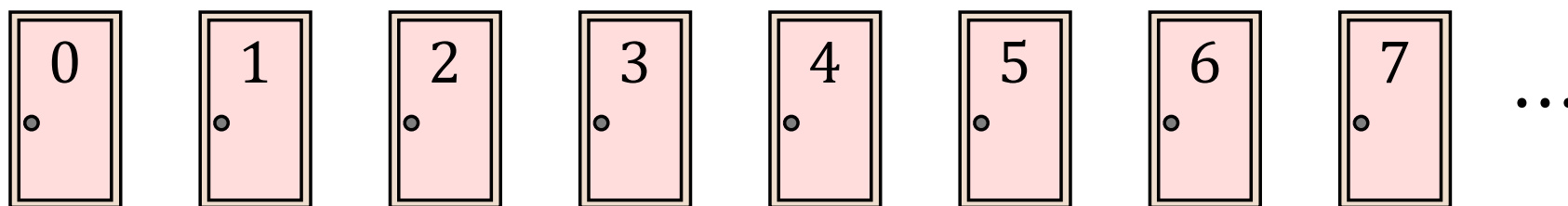


CS 483 - Introduction to the Theory of Computation

Lecture 13

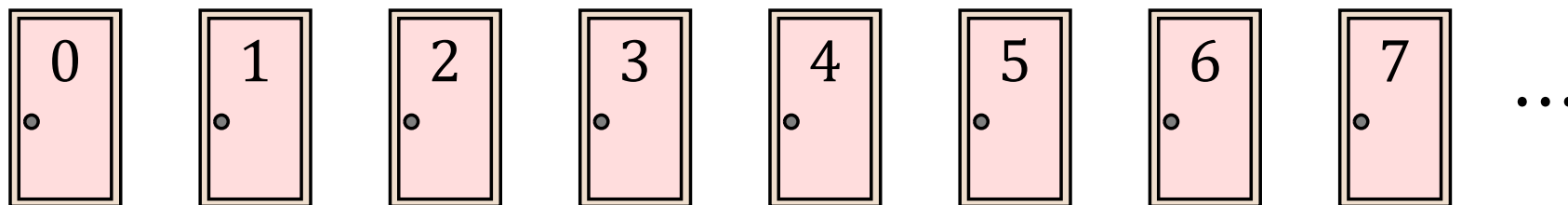
Hilbert's Hotel

Hilbert runs a hotel with infinitely many room, labeled with 0, 1, 2, 3, ...



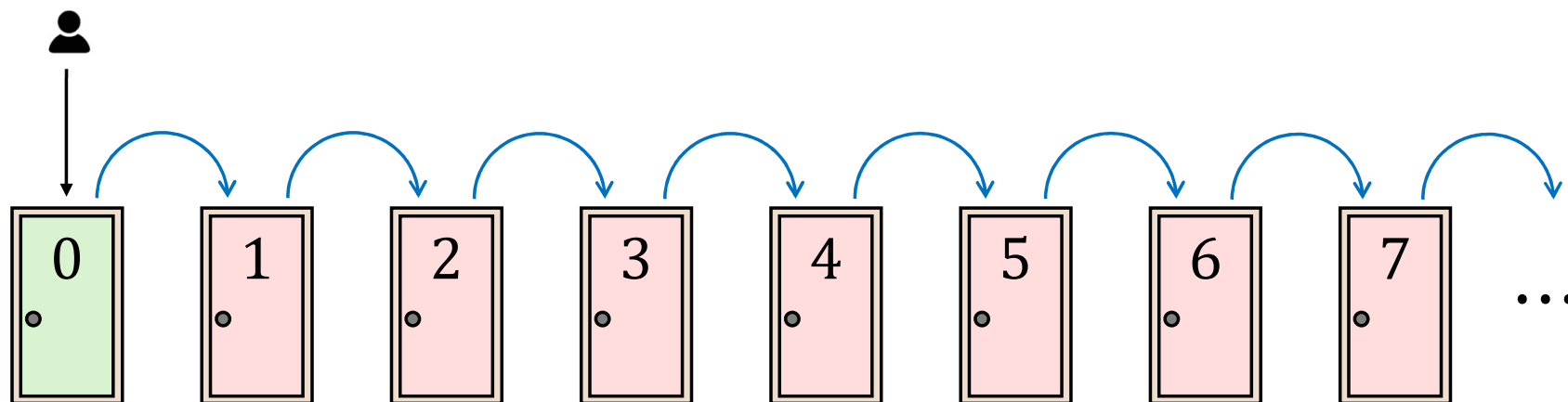
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The hotel is fully booked. One day a new guest shows up.



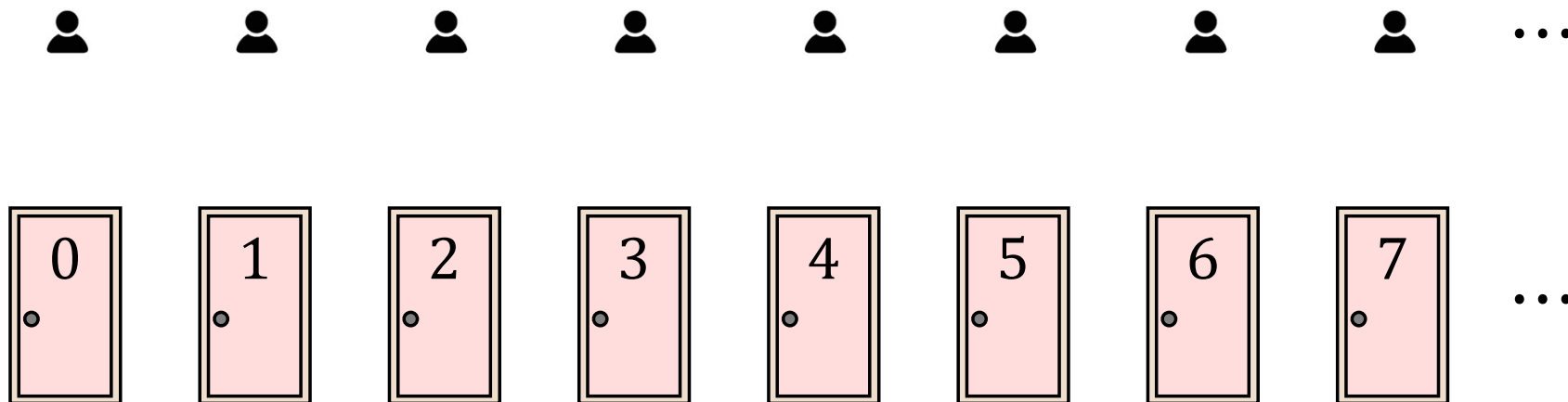
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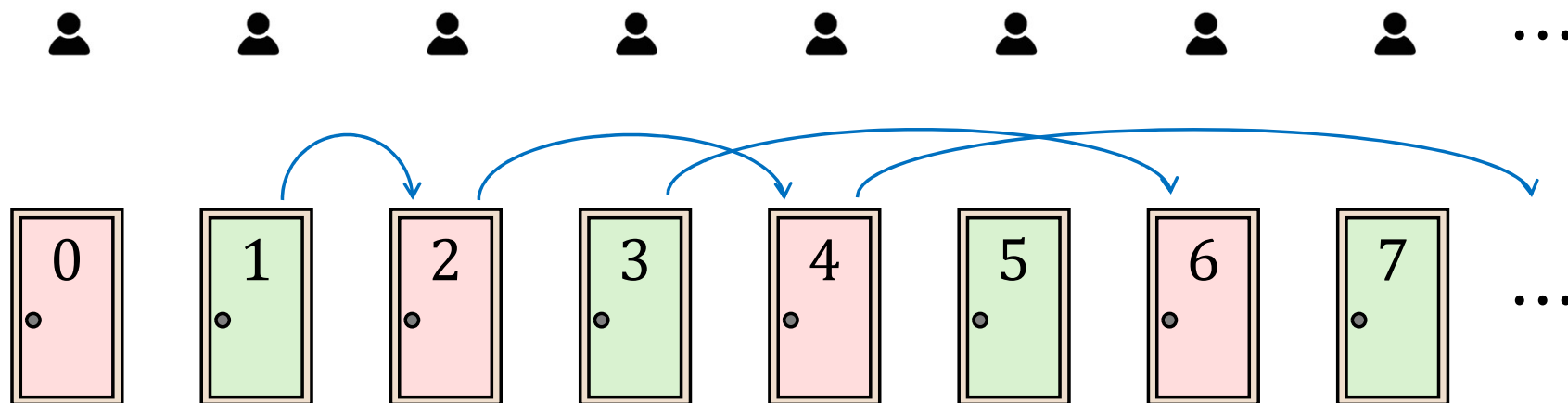
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The second day, every guest brings another new guest to the hotel.



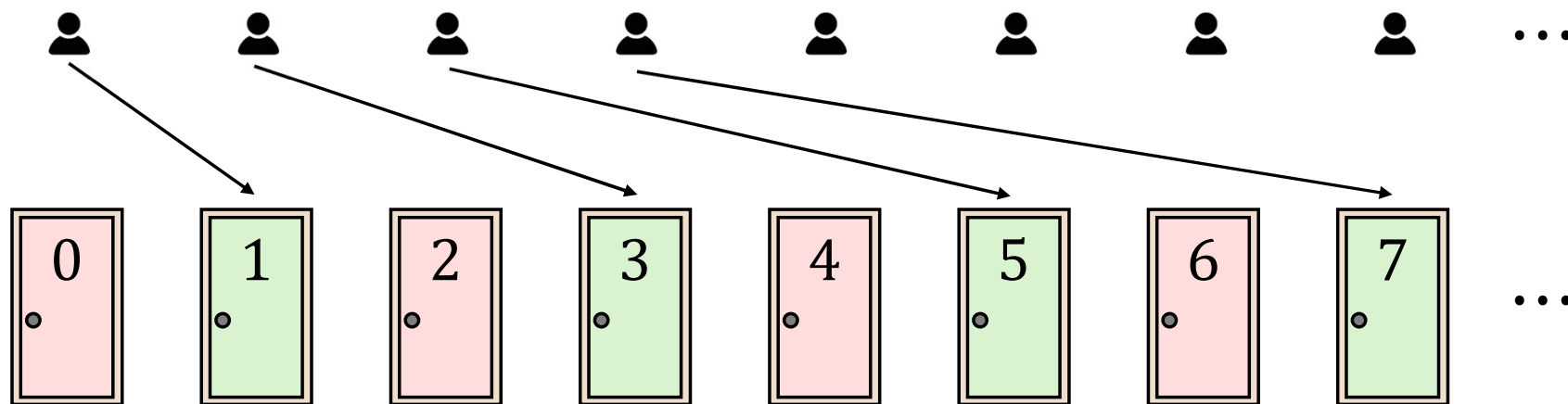
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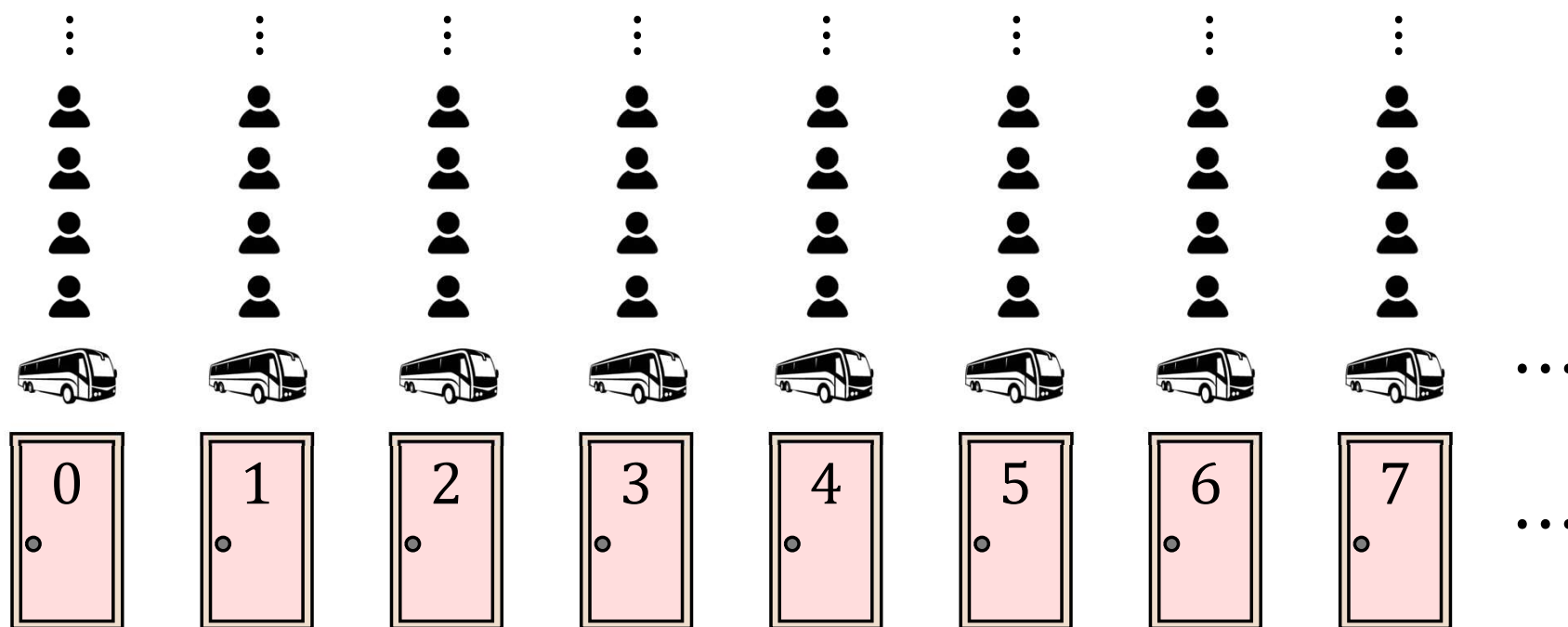
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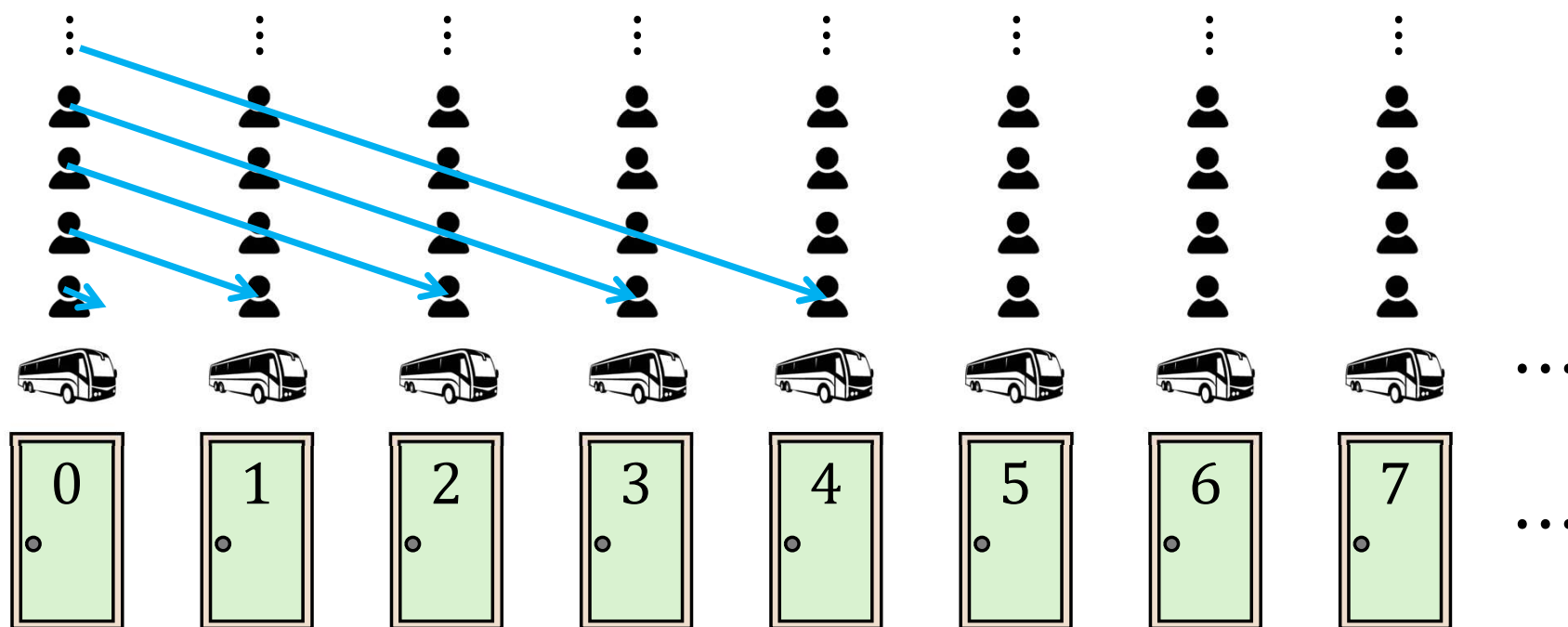
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The third day, every guest refers infinitely many new guests to the hotel.



Hilbert's Hotel

i -th guest on j -th bus: Room $\frac{1}{2}(i+j)(i+j+1) + i$



Countable Sets

A set A is **countably infinite**, if its elements can be written as an infinite sequence:

$$A = \{a_0, a_1, a_2, \dots\},$$

and $a_i \neq a_j$ whenever $i \neq j$.

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Note the difference with **enumerability**: the sequence for countability is **NOT necessarily** computable.

Countable Sets

What we showed via Hilbert's hotel:

- If both A and B are countable, then $A \cup B$ is countable.
- If both A and B are countable, then $A \times B$ is countable.

Countable Sets

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You may have seen equations like:

$$\infty + 1 = \infty, \infty + \infty = \infty, \infty \cdot \infty = \infty$$

Formally, these are true when ∞ represents an infinite cardinal number.

Set Cardinality

Two set A and B have the same **cardinality**, if there exists a bijection (one-to-one correspondence) between A and B .

Each equivalence class under same cardinality corresponds to a cardinal number. The cardinality of set A is denoted $|A|$.

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Let $\aleph_0 = |\mathbb{N}|$ (called “aleph zero”).

Then a set A is countably infinite if and only if $|A| = \aleph_0$.

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We say $|A| \leq |B|$, if there exists an injection from A to B .

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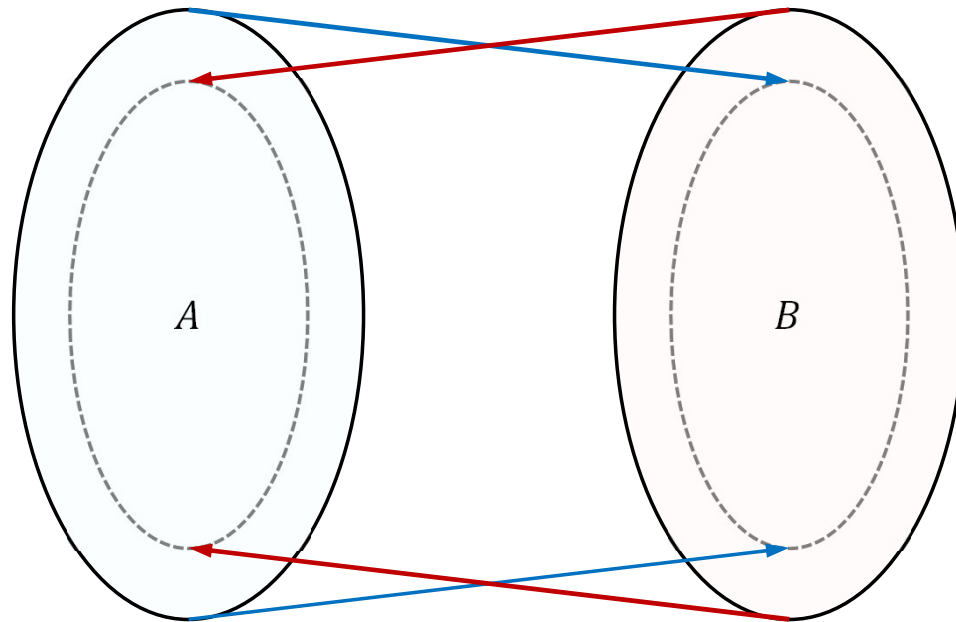
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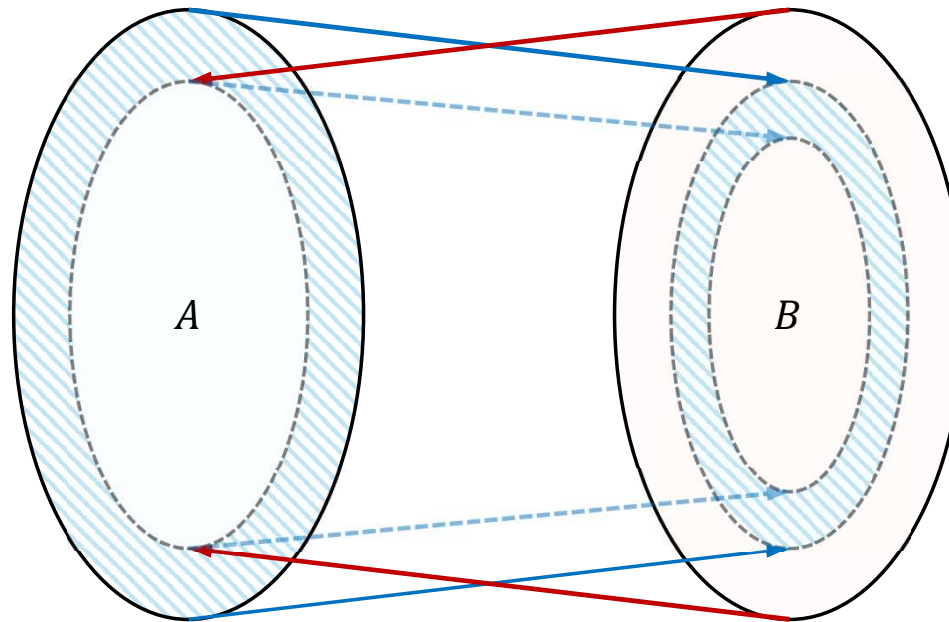
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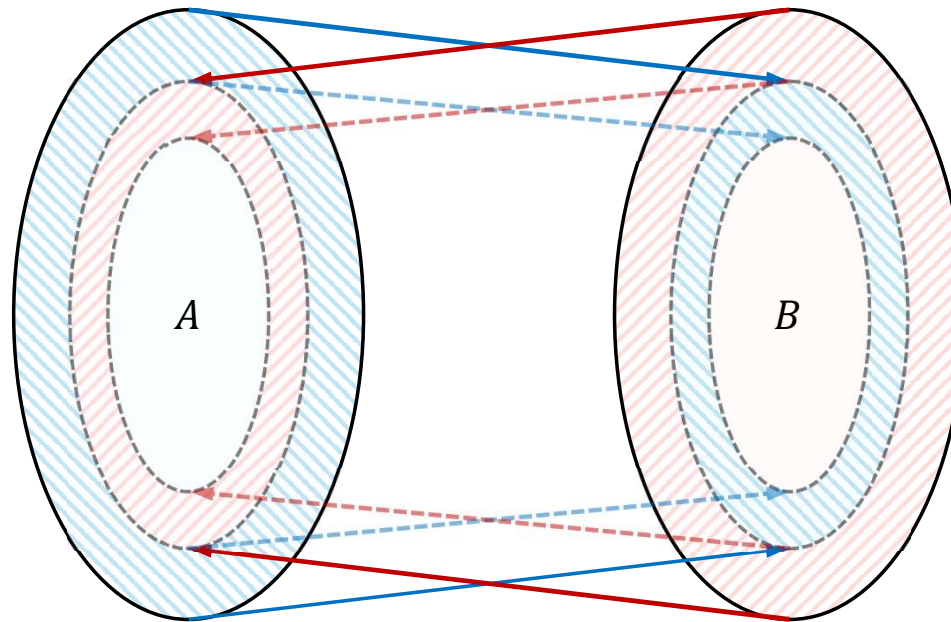
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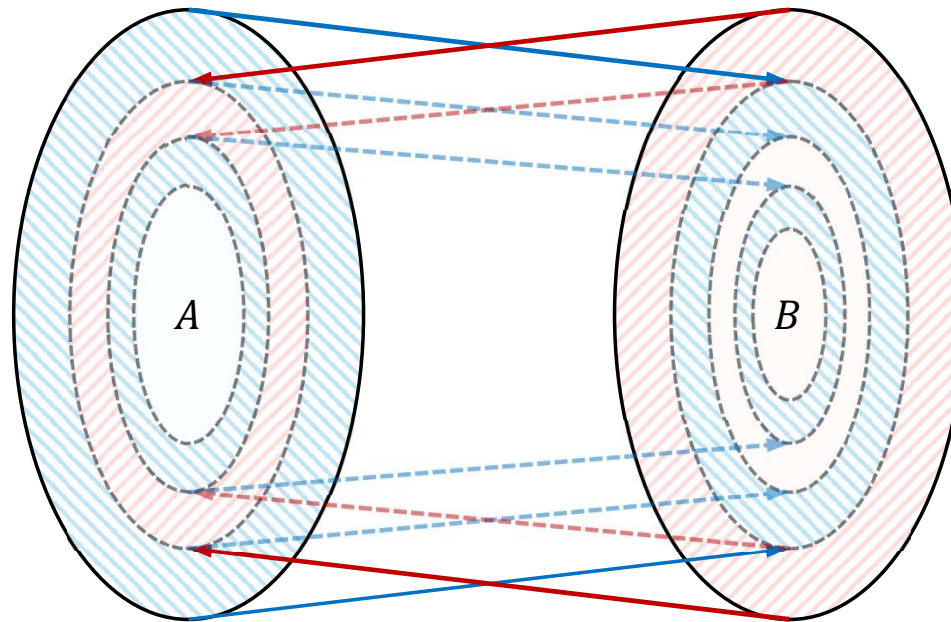
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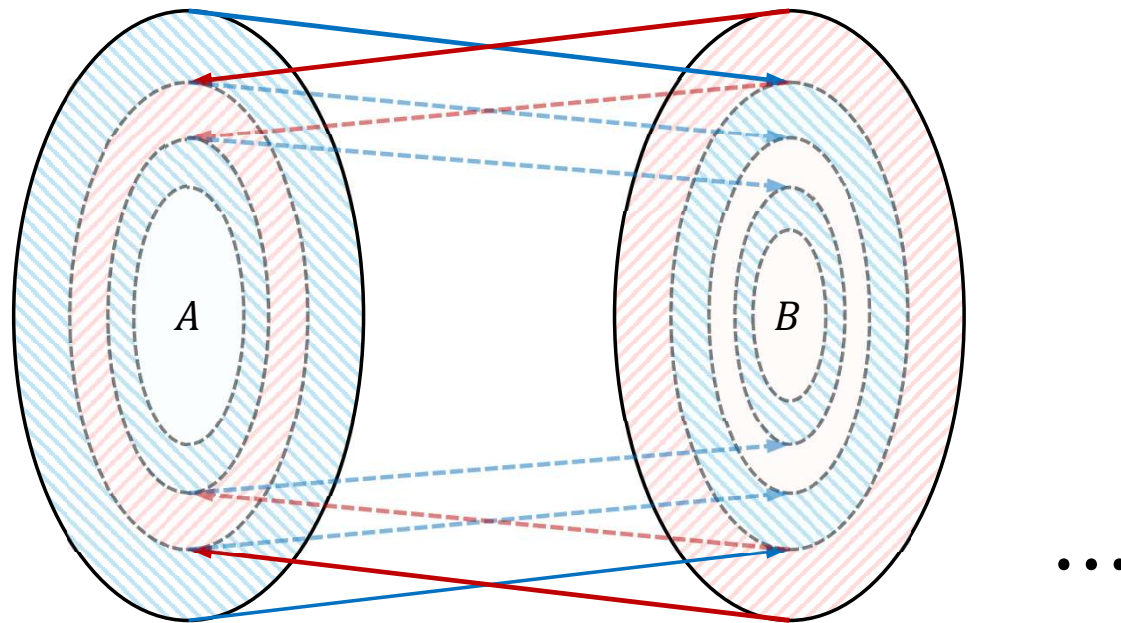
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Proof. Inductively define $f: \mathbb{N} \rightarrow A$. For each $i \in \mathbb{N}$, choose

$$f(i) \in A \setminus \{f(0), f(1), \dots, f(i-1)\}$$

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Corollary. If a set A is infinite and $|A| \leq \aleph_0$, then $|A| = \aleph_0$.

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The following sets are all countably infinite:

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and g is a surjection.

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Corollary. If $f: A \rightarrow B$ is surjective and $|A| = \aleph_0$, then $|B| \leq \aleph_0$.

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We say $|A| < |B|$, if $|A| \leq |B|$ and $|A| \neq |B|$.

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Set A is **uncountable**, if $\aleph_0 < |A|$.

Corollary. For every infinite set A , 2^A is uncountable.

Corollary. If A is infinite and $|2^A| \leq |B|$, then B is uncountable.

Barber Paradox

There is a town barber, who sticks to the following principle:

*I always give, and only give shaves
to those who don't shave themselves.*

Does the barber shave himself?



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- If $y \in S$, then by definition of S we have $y \notin f(y) = S$;
- If $y \notin S = f(y)$, then by definition of S we have $y \in S$.

Cantor's Theorem

This technique is called **diagonalization**.

	a	b	c	d	e	$\dots$
$f(a)$	✓	✓		✓		
$f(b)$			✓			
$f(c)$		✓	✓			
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- $\mathbb{R},$ via injection from $[0,1]: x \mapsto x$
- $\mathbb{R} \setminus \mathbb{Q},$ since $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}).$

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As a result, **RE** is **countable** since $L(M)$ is a surjection onto **RE**.

The following sets are all **countable**, none of which equals **ALL**:

- the set of regular languages on Σ ;
- the set of context-free languages on Σ ;
- **R**, **coRE**, and **RE $\cup$ coRE**.