CS 483 - Introduction to the Theory of Computation

Lecture 5

Non-regular languages

Is every language regular?

Short answer: No, because # of regular languages $<$ # of languages.

(We will see in future lectures what this exactly means.)

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Long answer: We will prove the following language

$$L = \{a^n b^n \mid n \in \mathbb{N}\}$$

is not regular.

Non-regular languages

Prove by contradiction. Suppose that L is computed by a DFA M.

$$r_0 \xrightarrow{a} r_1 \xrightarrow{a} \cdots \xrightarrow{a} r_n \xrightarrow{b} \cdots \xrightarrow{b} r_{2n}$$

Key intuition: M does not scale with n.

Non-regular languages

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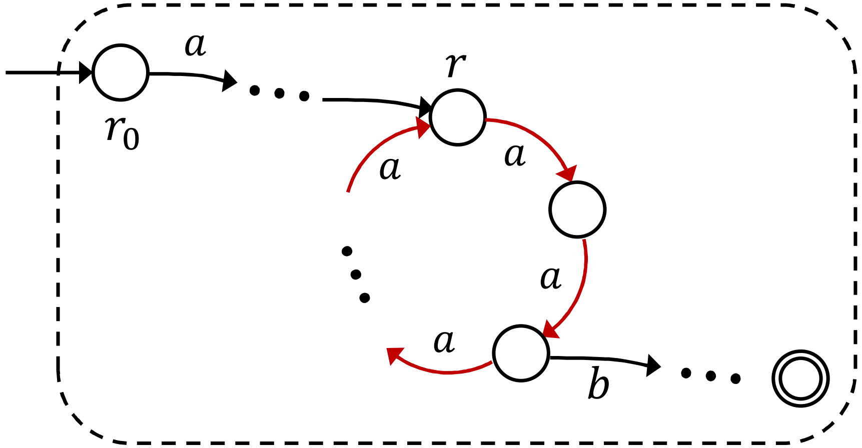
Key intuition: M does not scale with n.
For large enough n ($n \geq |Q|$),
$r_0, r_1, \ldots, r_n$ will contain repeated states.



Non-regular languages

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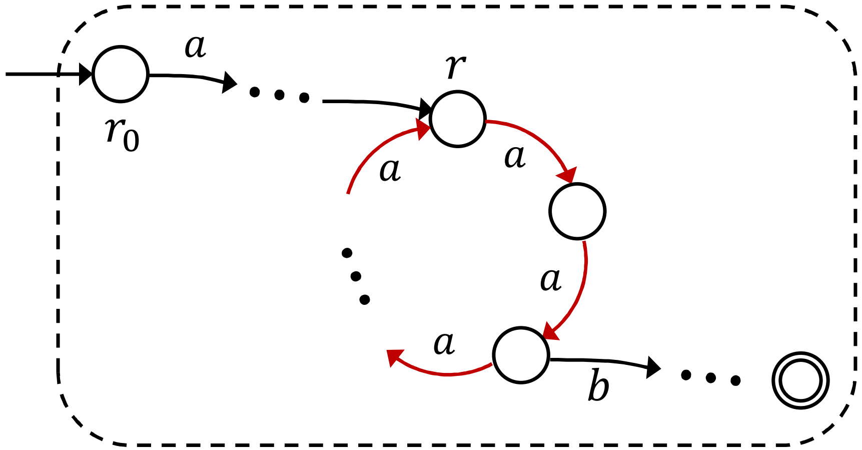

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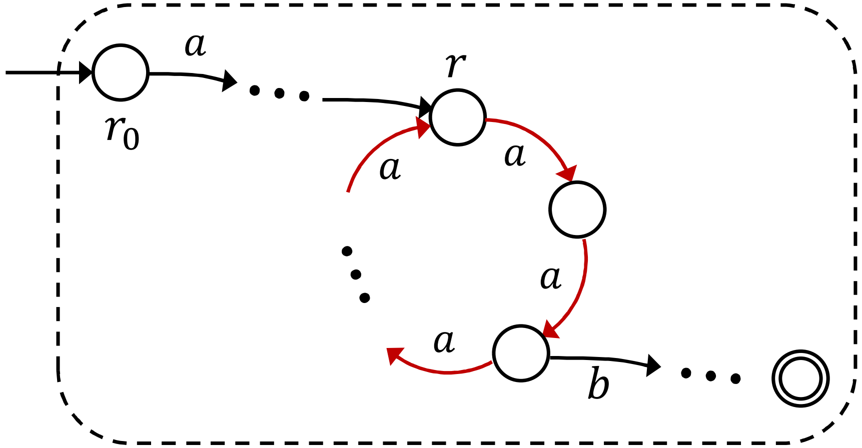
For large enough n ($n \geq |Q|$),
$r_0, r_1, \ldots, r_n$ will contain repeated states.

If $r_i = r_j$, $0 \leq i < j \leq n$, then
$a^{n+j-i} b^n \in L(M)$, contradiction.



Pumping Lemma

Theorem. Let L be a regular language. Then there exists $\ell \in \mathbb{N}_+$ (called pumping length) such that every $w \in L$ with $|w| \geq \ell$ can be written as $w = xyz$, satisfying:

- $|y| > 0$,
- $|xy| \leq \ell$,
- $xy^k z \in L$ for every $k \in \mathbb{N}$.

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I.e., the first ℓ symbols must contain a non-empty part that can be arbitrarily repeated (pumped).

Pumping Lemma

Proof. Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA that computes L. Let $\ell = |Q|$.

Given $w = w_1 \dots w_n \in L(M)$, let $r_0, r_1, \dots, r_n$ be the sequence of states of M on input w.

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Given $w = w_1 \dots w_n \in L(M)$, let $r_0, r_1, \dots, r_n$ be the sequence of states of M on input w.

By pigeonhole principle on $r_0, r_1, \dots, r_\ell$, there exists $0 \le i < j \le \ell$ such that $r_i = r_j$. Let

$$x = w_1 \dots w_i,\ y = w_{i+1} \dots w_j,\ z = w_{j+1} \dots w_n$$

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$$x = w_1 \dots w_i,\ y = w_{i+1} \dots w_j,\ z = w_{j+1} \dots w_n$$

The sequence of states on input $xy^k z$ will be $r_0, \dots r_i, \left(r_{i+1}, \dots, r_j\right)^k, \dots, r_n$.

Applications of Pumping Lemma

Example 1.

$$L_1 = \{a^n b^n \mid n \in \mathbb{N}\}$$

Assume L_1 is regular. Let ℓ be the pumping length. Take $w = a^\ell b^\ell \in L_1$ and let $w = xyz$ be the partition in the pumping lemma.

Applications of Pumping Lemma

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Assume L_1 is regular. Let ℓ be the pumping length. Take $w = a^\ell b^\ell \in L_1$ and let $w = xyz$ be the partition in the pumping lemma.

Since $|y| > 0$ and $|xy| \leq \ell$, it must be the case that $y = a^t$ with $t > 0$.

Then $xy^2z = a^{\ell+t} b^\ell \notin L_1$, a contradiction.

Applications of Pumping Lemma

Example 2.

$$L_2 = \{a^m b^n \mid m \geq n\}$$

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Assume L_2 is regular. Let ℓ be the pumping length. Take $w = a^\ell b^\ell \in L_2$ and let $w = xyz$ be the partition in the pumping lemma.

Since $|y| > 0$ and $|xy| \leq \ell$, it must be the case that $y = a^t$ with $t > 0$.

Then $xy^0 z = a^{\ell - t} b^\ell \notin L_2$, a contradiction.

Applications of Pumping Lemma

Example 3.

$$L_3 = \{w \in \{a,b\}^* \mid w \text{ contains same numbers of } a \text{ and } b\}$$

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Since $|y| > 0$ and $|xy| \leq \ell$, it must be the case that $y = a^t$ for $t > 0$.

Then $xy^2z = a^{\ell+t}b^\ell \notin L_3$, a contradiction.

Applications of Pumping Lemma

Example 3.

$$L_3 = \{w \in \{a, b\}^* \mid w \text{ contains same numbers of } a \text{ and } b\}$$

Alternative proof: $L_1 = L_3 \cap a^* b^*$.

Since $a^* b^*$ is regular, if L_3 is regular, then L_1 is also regular (PSet 1), a contradiction.

Applications of Pumping Lemma

Example 4.

$$L_4 = \{w \in \{a,b\}^* \mid w \text{ contains different numbers of } a \text{ and } b\}$$

Proof by closure property:

$L_4 = \overline{L_3}$. If L_4 is regular, then L_3 is also regular (PSet 1), a contradiction.

Applications of Pumping Lemma

Example 4.

$$L_4 = \{w \in \{a,b\}^* \mid w \text{ contains different numbers of } a \text{ and } b\}$$

Proof by pumping lemma:

Assume L_4 is regular. Let ℓ be the pumping length. Take $w = a^\ell b^{\ell+\ell!} \in L_4$ and let $w = xyz$ be the partition in the pumping lemma.

Since $|y| > 0$ and $|xy| \leq \ell$, it must be the case that $y = a^t$ for $t > 0$.

Then $xy^{1+\ell!/t}z = a^{\ell+\ell!}b^{\ell+\ell!} \notin L_4$, a contradiction.

Applications of Pumping Lemma

Example 5.

$$L_5 = \{ww \mid w \in \{a, b\}^*\}$$

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Assume L_5 is regular. Let ℓ be the pumping length. Take $w = a^\ell b a^\ell b \in L_5$ and let $w = xyz$ be the partition in the pumping lemma.

Since $|y| > 0$ and $|xy| \leq \ell$, it must be the case that $y = a^t$ for $t > 0$.

Then $xy^2z = a^{\ell+t} b a^\ell b \notin L_5$, a contradiction.

Applications of Pumping Lemma

Example 6.

$$L_6 = \{a^{n^2} \mid n \in \mathbb{N}\}$$

Applications of Pumping Lemma

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Assume L_6 is regular. Let ℓ be the pumping length. Take $w = a^{\ell^2} \in L_6$.

By the pumping lemma, $a^{\ell^2+t} \in L_6$ for some $0 < t \leq \ell$.

But $\ell^2 < \ell^2 + t < (\ell + 1)^2$, a contradiction.

Applications of Pumping Lemma

Example 7.

$$L_7 = \{a^n \mid n \in \mathbb{N} \text{ is a prime number}\}$$

Applications of Pumping Lemma

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Assume L_7 is regular. Let ℓ be the pumping length. Let $p \geq \ell$ be a prime number and take $w = a^p \in L_7$.

By the pumping lemma, $a^{p+pt} \in L_7$ for some $0 < t \leq \ell$.

But $p + pt = p(1 + t)$ is not a prime number, a contradiction.

Applications of Pumping Lemma

Example 8.

$$L_8 = \{\, c^m a^n b^n \mid m \geq 1, n \in \mathbb{N} \,\} \cup \{a^m b^n \mid m, n \in \mathbb{N}\}$$

Applications of Pumping Lemma

Example 8.

$$L_8 = \{\, c^m a^n b^n \mid m \geq 1, n \in \mathbb{N} \,\} \cup \{a^m b^n \mid m, n \in \mathbb{N}\}$$

Assume L_8 is regular. Let ℓ be the pumping length. Take $w = c^m a^n b^n \in L_8$ and let $w = xyz$ be the partition in the pumping lemma.

Applications of Pumping Lemma

Example 8.

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Assume L_8 is regular. Let ℓ be the pumping length. Take $w = c^m a^n b^n \in L_8$ and let $w = xyz$ be the partition in the pumping lemma.

It is always possible that $y = c^t$ with $0 < t \leq m$ and indeed

$$xy^k z = c^{m+(k-1)t} a^n b^n \in L_8.$$

Does it mean that L_8 is regular?

Applications of Pumping Lemma

Example 8.

$$L_8 = \{\, c^m a^n b^n \mid m \geq 1, n \in \mathbb{N} \,\} \cup \{a^m b^n \mid m, n \in \mathbb{N}\}$$

Assume L_8 is regular, then $L_8 \cap ca^*b^* = \{ca^n b^n \mid n \in \mathbb{N}\,\}$ is also regular.

Let M be the DFA that computes $L_8 \cap ca^*b^*$, and construct the ε-NFA M' by replacing every c-transition in M to an ε-transition.

Then $L(M') = \{\, a^n b^n \mid n \in \mathbb{N} \,\} = L_1$ which is not regular, a contradiction.

Generalized Pumping Lemma

Theorem. Let L be a regular language. Then there exists $\ell \in \mathbb{N}_+$ (called pumping length) such that every $vw \in L$ with $|w| \geq \ell$ can be written as $vw = vxyz$, satisfying:

- $|y| > 0$,
- $|xy| \leq \ell$,
- $vxy^k z \in L$ for every $k \in \mathbb{N}$.