

CS 483 - Introduction to the Theory of Computation

Lecture 9

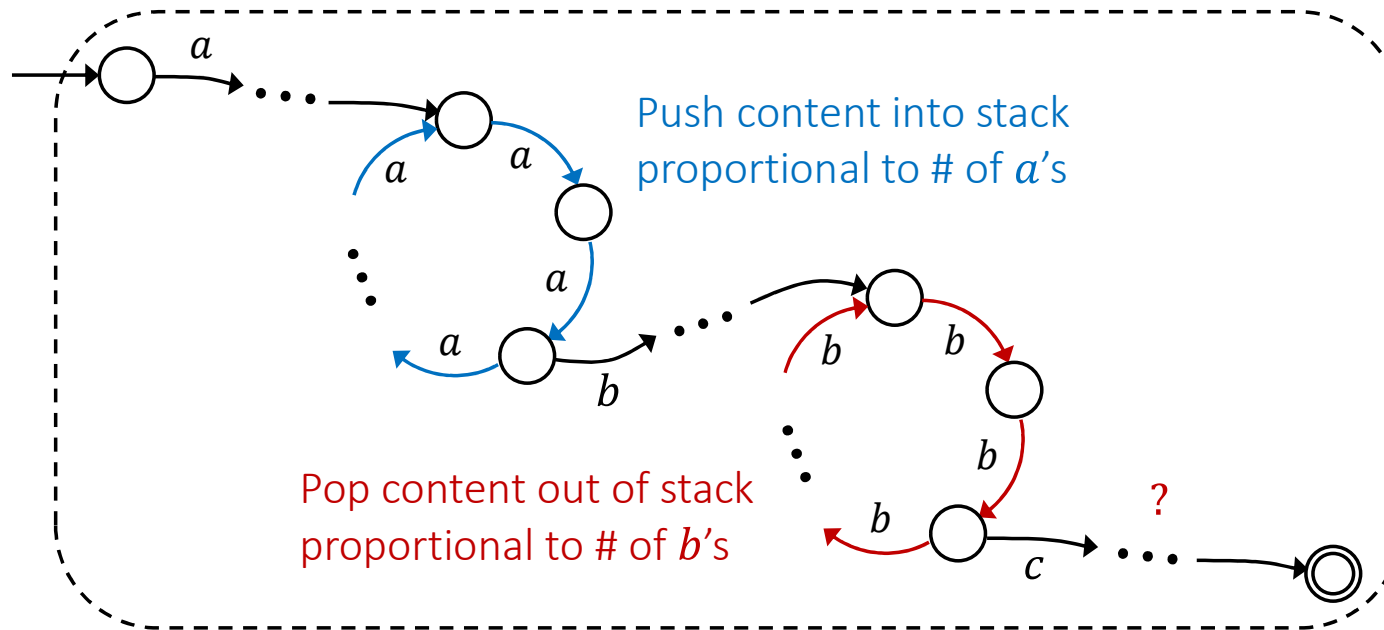
Non-Context-Free Languages

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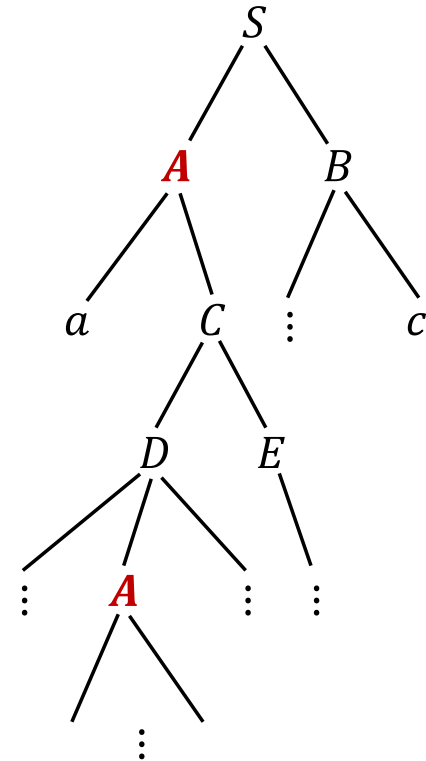
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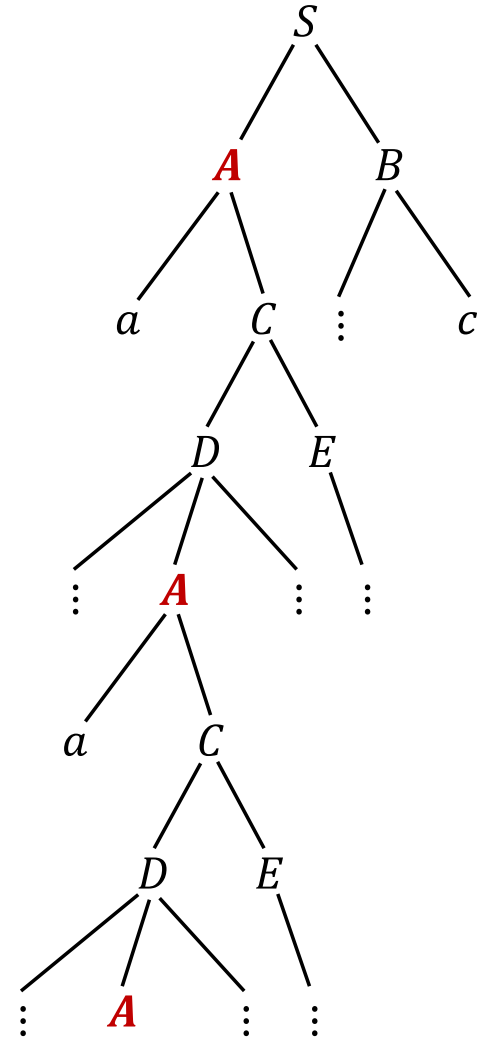
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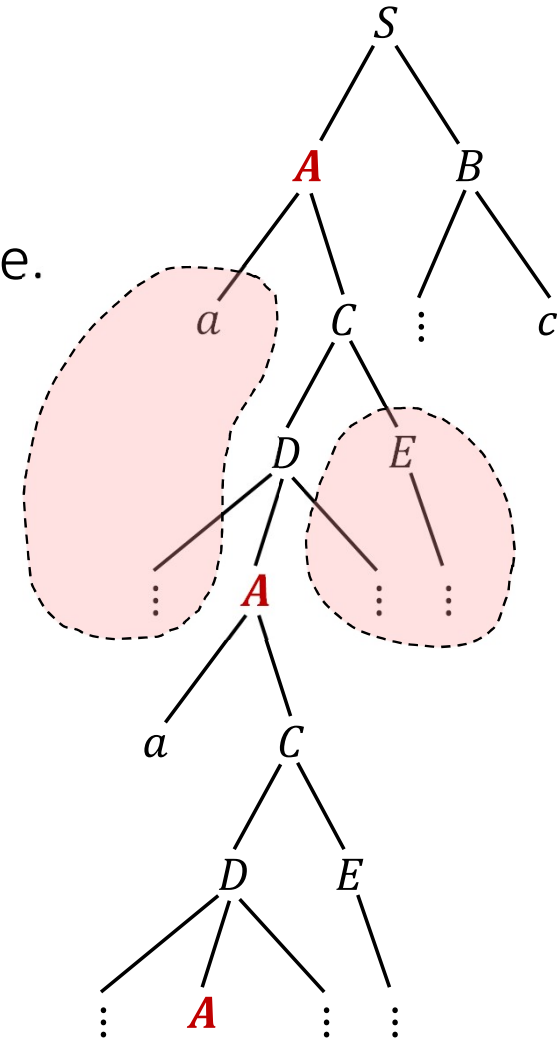
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Pumping Lemma for CFL

Theorem. Let L be a CFL. Then there exists $\ell \in \mathbb{N}_+$ (called **pumping length**) such that every $w \in L$ with $|w| \geq \ell$ can be written as $w = uvxyz$, satisfying:

- $|v| + |y| > 0$,
- $|vxy| \leq \ell$,
- $uv^kxy^kz \in L$ for every $k \in \mathbb{N}$.

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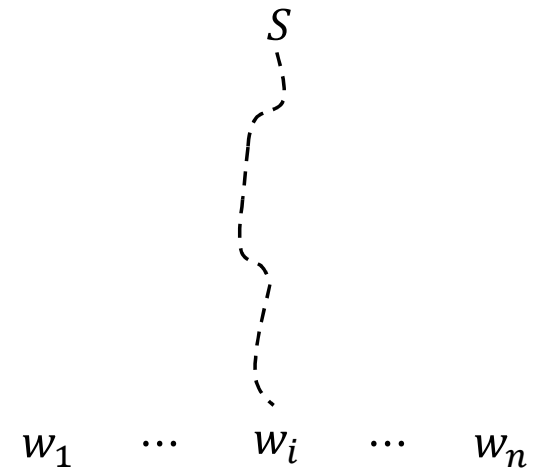
- $|v| + |y| > 0$,
- $|vxy| \leq \ell$,
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I.e., there must be two parts within ℓ symbols that can be arbitrarily **simultaneously** repeated (pumped).

Pumping Lemma for CFL

Proof. Let $G = (\Sigma, N, P, S)$ be the CFG for L under Chomsky normal form, and let $\ell = 2^{|N|+1}$.

Given $w = w_1 \dots w_n \in L$ with $n \geq \ell$, suppose w_i is the deepest node in the parsing tree. Since depth- d parsing tree contains at most 2^d leaves, the depth of w_i is at least $|N| + 1$.

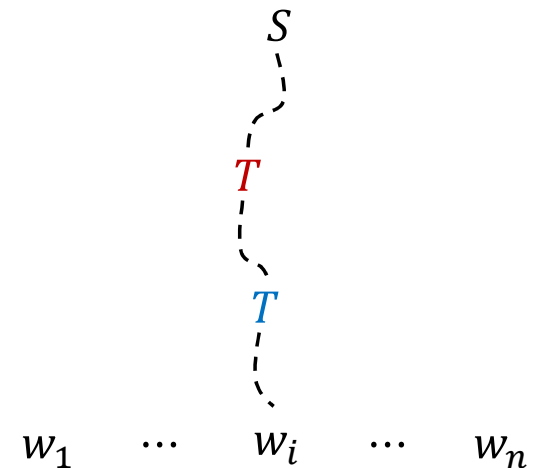


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Consider the last $|N| + 1$ non-terminal symbols on the path from root to w_i , there must be a $T \in N$ that appears at least twice.



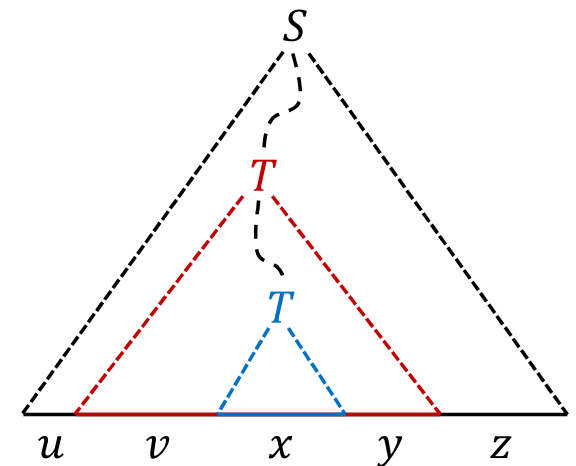
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We have $T \xRightarrow{*} vTy$ and $T \xRightarrow{*} x$, so $T \xRightarrow{*} v^kxy^k$.



Applications of Pumping Lemma

Example 1.

$$L_1 = \{a^n b^n c^n \mid n \in \mathbb{N}\}$$

Assume L_1 is a CFL. Let ℓ be the pumping length. Take $w = a^\ell b^\ell c^\ell \in L_1$ and let $w = uvxyz$ be the partition in the pumping lemma.

Applications of Pumping Lemma

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Assume L_1 is a CFL. Let ℓ be the pumping length. Take $w = a^\ell b^\ell c^\ell \in L_1$ and let $w = uvxyz$ be the partition in the pumping lemma.

Since $|vxy| \leq \ell$, it is either contained in $a^\ell b^\ell$ or in $b^\ell c^\ell$:

- If vxy is in $a^\ell b^\ell$, the number of c in uv^2xy^2z is smaller than a or b ;
- If vxy is in $b^\ell c^\ell$, the number of a in uv^2xy^2z is smaller than b or c .

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Example 2.

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Since $|vxy| \leq \ell$, it is either contained in $a^\ell b^\ell$ or in $b^\ell c^\ell$:

- If vxy is in $a^\ell b^\ell$, the number of c in uv^2xy^2z is smaller than a or b ;
- If vxy is in $b^\ell c^\ell$, the number of a in uxz is larger than b or c .

Applications of Pumping Lemma

Example 3.

$$L_3 = \{w \in \{a, b, c\}^* \mid w \text{ contains same number of } a, b, c\}$$

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Applications of Pumping Lemma

Example 4.

$$L_4 = \{ww \mid w \in \{a, b\}^*\}$$

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Assume L_4 is a CFL. Let ℓ be the pumping length. Take $w = a^\ell b^\ell a^\ell b^\ell \in L_4$ and let $w = uvxyz$ be the partition in the pumping lemma.

Suppose $uv^2xy^2z = w'w'$. w' must start with a^ℓ and end with b^ℓ . So

$$uv^2xy^2z = a^\ell s b^\ell a^\ell s b^\ell$$

for some $s \in \{a, b\}^*$, $|s| > 0$, which contradicts $|vxy| \leq \ell$.

Applications of Pumping Lemma

Example 5.

$$L_5 = \{a^{n^2} \mid n \in \mathbb{N}\}$$

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Assume L_5 is a CFL. Let ℓ be the pumping length. Take $w = a^{\ell^2} \in L_5$.

By the pumping lemma, $a^{\ell^2+t} \in L_5$ for some $0 < t \leq \ell$.

But $\ell^2 < \ell^2 + t < (\ell + 1)^2$, a contradiction.

Applications of Pumping Lemma

Example 6.

$$L_6 = \{d^m a^n b^n c^n \mid m \geq 1, n \in \mathbb{N}\} \cup \{a^m b^n c^p \mid m, n, p \in \mathbb{N}\}$$

Just like the example for regular language, L_6 can be pumped.

Does it mean that L_6 is context-free?

Generalized Pumping Lemma for CFL

Theorem (Ogden's lemma). Let L be a CFL. Then there exists $\ell \in \mathbb{N}_+$ such that every $w \in L$ with at least ℓ **marked symbols** can be written as $w = uvxyz$, satisfying:

- vy contains at least one marked symbol,
- vxy contains at most ℓ marked symbols,
- $uv^kxy^kz \in L$ for every $k \in \mathbb{N}$.

Applications of Ogden's lemma

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Assume L_6 is a CFL. Let ℓ be the pumping length. Take $w = da^{\ell+1}b^{\ell+1}c^{\ell+1} \in L_6$, mark $b^{\ell+1}$ and let $w = uvxyz$ be the partition in Ogden's lemma.

Applications of Ogden's lemma

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Since vxy contains at most ℓ marked symbols, it can't contain both a and c :

- If vxy is in $da^{\ell+1}b^{\ell+1}$, the number of c in uv^2xy^2z is smaller than b ;
- If vxy is in $b^{\ell+1}c^{\ell+1}$, the number of a in uv^2xy^2z is smaller than b .

Closure Properties of CFL

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- intersection ✗ $L_1 = \{a^n b^n c^m\} \cap \{a^m b^n c^n\}$
- complement ✗ De Morgan's Law:

$$A \cap B = \overline{\overline{A} \cup \overline{B}}$$

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Theorem. If A is context-free and B is **regular**, then $A \cap B$ is context-free.

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Proof. Let $M_1 = (Q_1, \Sigma, \Gamma, \delta_1, q'_0, F_1)$ be a PDA for A , and $M_2 = (Q_2, \Sigma, \delta_2, q''_0, F_2)$ be an ε -NFA for B . Assume M_1 and M_2 contains all the trivial transitions:

$$(q_1, \varepsilon) \in \delta_1(q_1, \varepsilon, \varepsilon), \quad q_2 \in \delta_2(q_2, \varepsilon)$$

We construct PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ where:

- $Q = Q_1 \times Q_2, q_0 = (q'_0, q''_0), F = F_1 \times F_2$
- $\delta((q_1, q_2), a, s) = \{((q'_1, q'_2), t) \mid (q'_1, t) \in \delta_1(q_1, a, s), q'_2 \in \delta_2(q_2, a)\}$

Applications of Pumping Lemma

Example 6.

$$L_6 = \{d^m a^n b^n c^n \mid m \geq 1, n \in \mathbb{N}\} \cup \{a^m b^n c^p \mid m, n, p \in \mathbb{N}\}$$

Assume L_6 is a CFL. Then

$$L'_6 = L_6 \cup da^*b^*c^* = \{da^n b^n c^n \mid n \in \mathbb{N}\}$$

is also a CFL. Change d -transitions in the PDA for L'_6 to ε -transitions and we get a PDA for L_1 , a contradiction.

Chomsky Hierarchy

