CS58400 - Theory of Computation and Computational Complexity Spring 2026

Assignment 1 *Due: Feb 19*

Note: Use LaTeX to typeset your solutions. You can use the source code of this file as a template or reference. Bonus questions are still capped by the total assignment grades, so only work on them if you want a challenge.

Problem 1 (Busy beavers). Consider Turing machines on the binary alphabet $\{0, 1\}$, with one single tape (no input or output tapes) that is infinite on both directions. For each $n \in \mathbb{Z}_+$, define the *busy beaver number* $BB(n)$ as the maximum number of steps that such a machine with n states halts in.

> **(10 pts)**. Show that $BB(n)$ cannot be bounded by any computable function: that is, for any computable function $f : \mathbb{Z}_+ \to \mathbb{Z}_+$, it cannot hold that $BB(n) = O(f(n))$.

Problem 2 (Very sparse language). A language $L \subseteq \{0, 1\}^*$ is called *very sparse*, if

$$\forall n \in \mathbb{N}, \quad |L \cap \{0, 1\}^n| \leq 1.$$

In other words, for each n, there is at most one length-n string in L.

1. **(5 pts)**. Show that there exists a very sparse language in $\mathsf{RE} \setminus \mathsf{R}$.
2. **(5 pts)**. Prove that if a very sparse language L satisfies

$$\forall n \in \mathbb{N}, \quad |L \cap \{0, 1\}^n| = 1.$$

   and $L \in \mathsf{RE}$, then $L \in \mathsf{R}$.

Problem 3 (Closure under taking prefixes). Consider the operator that takes all the prefixes (including the empty string and the string itself) of every element in a language $L \subseteq \{0, 1\}^*$:

$$\mathrm{pref}(L) = \big\{x \mid \exists y \in \{0, 1\}^*, xy \in L\big\}.$$

1. **(5 pts)**. Prove that RE is closed under the taking all prefixes. That is, if $L \in \mathsf{RE}$, then $\mathrm{pref}(L) \in \mathsf{RE}$.
2. **(5 pts)**. Prove that R is not closed under taking all prefixes.

   Hint. Think of how $\mathrm{pref}(L)$ can hide information that is useful for deciding L.

Problem 4 (Completeness in the arithmetic hierarchy). For a Turing machine M, we use $L(M) = \{x \in \{0,1\}^* \mid M(x) = 1\}$ to denote the language it recognizes.

1. **(5 pts)**. Prove that the following language:

$$\textsc{Emp} = \{\langle M\rangle \mid L(M) = \varnothing\}$$

   is $\mathsf{\Pi}_1 = \mathsf{coRE}$-complete under many-one reductions.

2. **(5 pts)**. Prove that the following language:

$$\textsc{Fin} = \{\langle M\rangle \mid L(M) \text{ is finite}\}$$

   is $\mathsf{\Sigma}_2$-complete under many-one reductions.

   Hint. Use Post's theorem to characterize $\mathsf{\Sigma}_2$. Notice that $L \subseteq \{0,1\}^*$ is finite if and only if in lexicographic order, L does not contain any $y > x$ for some x.

3. **(Bonus, 5 pts)**. Prove that there is no $\mathsf{\Delta}_n$-complete language under many-one reductions, for any $n > 1$.

   Hint. Each reduction is computed by a Turing machine. Use diagonalization.